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Cambridge Pre-U

Mathematics

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To Susie, as always.

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Introduction

This book has been designed to be a self-contained text which covers all the material, both Pure and Applied, required for the Cambridge Pre-U course in Mathematics (Principal) 9794. It includes the most recent adjustments to the specification, introduced for examination from 2016. Hitherto, no single text has fully covered all the required material in the desired manner.

This book has been prepared in four Parts, covering (in order) the Pure Mathematics, Mechanics and Probability aspects of the course, followed by Problem-solving. One of the important facets of a linear course, such as the Pre-U, is the interconnectivity of the material. This does lead to a design problem for a textbook, since there is no clear order in which the material must be presented; increasingly, different topics in the course rely on each other. The order of presentation of the Pure Mathematics material largely follows the order used in the Mathematics Department at Rugby School and develops the material progressively. The final three chapters in that Part, on Vectors, Complex Numbers and Numerical Methods are (to a larger extent) free-standing, and can be taught earlier in the course than their position in the Part might suggest, particularly the chapter on Numerical Methods. The Part on Mechanics does depend on the student having an understanding of vectors and a degree of sophistication with calculus, and so probably shouldn't be presented at the start of the course. The Part on Probability is self-contained to a large degree, and can be presented early on in a course of study, with the possible exception of the chapter on Permutations and Combinations. Experience shows that, while there are few facts in this topic, their application to problem-solving asks for a degree of sophistication from the student that is more likely to be found during the second half of the course.

At various stages in the book, attention is drawn to facts and information that are either crucial to an understanding of the topic, or else merely interesting, or in the nature of extension material. The different types of information is presented in boxes:

- **Key Facts** The information in these boxes is crucial, and students would be well-advised to learn them!

> **Key Fact 3.1** The Gauss Sum
>
> The formula for the sum of the first n integers is
>
> $$1+2+3+\cdots+n = \sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

- **Food for Thought** While not essential to the course, information provided in these boxes is related to the material at hand, and should be informative to the interested reader.

Food For Thought 3.1

Unlike GPs, APs do not have a sum to infinity. The infinite Gauss sum

$$1+2+3+4+\cdots = \sum_{r=1}^{\infty} r$$

has no limit, since the so-called partial sums

$$1+2+3+\cdots+n = \tfrac{1}{2}n(n+1)$$

get bigger without limit as n increases.

- **For Interest** The information in these boxes is provided as a matter of interest, and is not essential to an understanding of the course. The enthusiastic reader should find these boxes interesting, but they can be omitted if desired.

For Interest

There are many pages on the Internet that will tell you that

$$1+2+3+\cdots = \sum_{r=1}^{\infty} r = -\tfrac{1}{12}$$

These should be treated with caution. They contain arguments which are similar to

Since $\infty+1 = \infty = \infty+0$, we deduce that $1 = 0$.

What is bizarre is that many interesting results in Mathematical Physics can be deduced by using exactly this 'identity'. The solution to this apparent weirdness can be found in the details of a famous function call the **Riemann zeta function**.

Finally, any section which is particularly difficult, or slightly beyond the syllabus, is indicated by a bar in the margin, as shown here. I make no apologies for adding extra and demanding material to this book. The aim of the Pre-U is to encourage mathematical thought and problem-solving, and the aim of these extra entries is to show the interested student 'what happens next'.

Problem-solving is a particular feature of the Pre-U — the 'D1 tie-breaker' questions currently at the end of each paper require a degree of mental flexibility. The fourth Part, and last chapter, of this book contains a number of problems, and discusses a variety of solution techniques for each. It is important that the student develops an awareness of the possibility of there being more than one solution to a problem, so that they have the ability to handle an unusual question which might require a different technique to solve than the more standard questions.

Exercises are given at all stages of the book, and each set of questions provides a range of difficulty to train all students and to challenge the more able. For summary purposes, Revision Exercise sections are included every three chapters, again summarising the techniques learnt in each set of three chapters at a variety of levels. At all stages, harder questions are marked with an asterisk ⋆. Answers to all questions (where an answer is appropriate) are provided in the Appendix.

The Cambridge Sixth Term Examination Paper, being used as it is for admission to Mathematics courses at a number of Universities — Cambridge in particular — provides an excellent set of problems which stretch the student beyond the usual level of difficulty required by Pre-U questions. A number of STEP questions are included in each set of Revision Exercises, roughly one question per chapter of the book. To help (and encourage!) the student, full solutions to these STEP questions are given in Chapter A2 in the Appendix.

The List of Formulae MF20 is the official list of formulae for both the Pre-U Mathematics and Further Mathematics courses. Since it covers both courses, it necessarily contains much information that is not needed for the Single Mathematics qualification, which is the target of this book. The final Appendix of this book contains a cut-down version of List MF20, and contains only those formulae which are relevant to the Single Mathematics Pre-U. The fact that this cut-down list is half the size of the full list reinforces the need for students to be familiar with these formulae; finding the correct formula amongst a list containing twice as many results as needed is a challenge made unnecessary if these formulae have already been learned!

Mark Hennings,
Rugby School,
June 2016.

Part 1

Pure Mathematics

1

Surds and Indices

In this chapter we will learn:

- how to manipulate expressions involving surds,
- how to manipulate expressions involving indices.

1.1 Types of Number

Modern Mathematics is built on the back of thousands of years of mathematical thought. Over the centuries, mathematicians saw the need for ever more complicated ideas of number. It is still important nowadays to be aware of the hierarchy of number types, since different mathematical ideas and argumchents can be applied at different levels. We start by setting out the fundamental different types of number that we will encounter:

- The most fundamental numbers are those used for counting: the positive whole numbers $1, 2, 3, 4, \ldots$ These are called the **natural numbers**. The set of all natural numbers is denoted by the special symbol $\mathbb{N}$, so that

$$\mathbb{N} = \left\{1, 2, 3, 4, \ldots\right\}$$

- The natural numbers are sufficient to count (sheep, coins, etc.), and can be used to add, but they are insufficient if we want to be able to subtract (as Alice told the Red Queen, 'nine from eight I can't, you know'). To be able to do subtraction neatly, the number zero and negative whole numbers were introduced, giving us the integers: $\ldots, -3, -1, 0, 1, 2, 3, \ldots$. The set of all the integers is denoted by the special symbol $\mathbb{Z}$ ('Z' for *Zahl*, the German for 'number'), so that

$$\mathbb{Z} = \left\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\right\}$$

  The integers are a very important set of numbers. As well as being able to add and subtract integers, multiplication is possible, as is factorisation into primes. Studying the properties of the integers has generated some of the richest areas of modern mathematics.

- The **rational** numbers are those which can be expressed as fractions of integers in the form $\frac{p}{q}$, where p and q are integers, and $q \neq 0$. The set of rational numbers is denoted by the special symbol $\mathbb{Q}$ ('Q' for *quotient*), so that

$$\mathbb{Q} = \left\{\frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0\right\}$$

For Interest

We are using a standard notation to describe sets here. A set can be written in the form

$$\left\{x \,\middle|\, A\right\}$$

where x is an expression for a number in the set, and A is a condition, or set of conditions, that specify the types of number that are permitted. The vertical bar | (sometimes a colon is used) should be read as 'such that'. Thus the formula for $\mathbb{Q}$ given above can be read as 'the set of numbers $\frac{p}{q}$ such that p and q are integers where q is non-zero'.

- Not all numbers are rational, however. Important numbers, like $\sqrt{2}$ and π, cannot be written as fractions. Numbers that cannot be expressed as fractions are called **irrational** numbers, and the irrational and rational numbers together form the **real** numbers. The collection of all real numbers is denoted by $\mathbb{R}$. Actually what is meant by a number here is quite a difficult question: integers and rationals have a fairly concrete existence which is founded in our experience, but irrational numbers are more elusive. The Pythagorean schools of mathematics in Ancient Greece thought that all numbers should be fractions, and that numbers which were not fractions were irrational in both senses of the word! We will have to be content with thinking that numbers are quantities that can have a place found for them along a number line.

- Eventually, we will want to move off the number line and study numbers that do things that real numbers cannot. In particular, we will want to introduce the square root of -1, denoted i. Numbers of the form $a + ib$, where a, b are real, are called **complex** numbers, and the set of all complex numbers is denoted $\mathbb{C}$.

It is worth observing that:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}\,.$$

Each of our special sets contains all the preceding special sets as a subset. A Venn diagram for these sets would be five concentric circles!

When we cannot express a number as a fraction, we try to express it in decimals. Rational numbers either have decimal expansions which terminate

$$\tfrac{7}{10} = 0.7 \qquad \tfrac{3}{16} = 0.1875 \qquad \tfrac{11}{20} = 0.55$$

or they have recurring decimal expansions, i.e. ones which eventually start repeating in a regular pattern:

$$\tfrac{3}{11} = 0.\dot{2}\dot{7} = 0.27272727\ldots \qquad \tfrac{8}{15} = 0.5\dot{3} = 0.533333\ldots$$

$$\tfrac{7}{17} = 0.\dot{4}11764705882352\dot{9} = 0.41176470588235294 1176\ldots$$

The converse is true: any terminating or recurring decimal describes a rational number. It is therefore easy to write down irrational numbers, by constructing decimals which definitely do not recur:

$$0.1010010001000010000010000000100\ldots$$

but it is more interesting to be able to find out whether particular numbers are irrational or not.

Example 1.1.1. *Show that $\sqrt{2}$ i.e. the square root of 2 is irrational.*

Suppose that $\sqrt{2}$ was rational. Then we could write $\sqrt{2} = \frac{a}{b}$ as a fraction. We can assume that the fraction is in its lowest terms, so that the positive integers a, b have no common factor. Squaring the formula for $\sqrt{2}$ and multiplying by b^2 gives

$$a^2 = 2b^2\,,$$

and hence a^2 is an even integer. But this can only happen when a is even. Thus $a = 2c$ for some integer c. But then $2b^2 = (2c)^2 = 4c^2$, and hence

$$b^2 = 2c^2\,.$$

But this implies that b^2 is even, and so b is even.

We have come to the conclusion that a and b, which have no common factor, are both even, and hence both divisible by 2. The only way out of this impasse is to deduce that our original idea, that $\sqrt{2}$ was rational, is not true. Thus we deduce that $\sqrt{2}$ is irrational.

This is an example of an important method of argument: *Proof by Contradiction*. If assuming a fact leads to nonsense, we may deduce that our original assumption was incorrect.

EXERCISE 1A

1. It is easy to 'place' a fraction on the number line. For example, $4\frac{2}{3}$ is two-thirds of the way from 4 to 5. How can we be sure about where to place $\sqrt{2}$? Can we be sure it exists? Were the ancient Greeks right to be worried?

2. Do there exist real numbers which possess two or more different decimal expansions? If so, which?

3. Express $0.\dot{1}2\dot{3}$ and $0.2\dot{2}\dot{7}$ as fractions.

4. If a real number x has a recurring decimal expansion which comprises a sequence of n repeated digits (so that $n = 3$ for $x = 0.1\dot{2}8\dot{5}$), show that $(10^n - 1)x$ has a terminating decimal expansion, and hence that x is rational.

5*. How many remainders are possible when dividing an integer by 17? Show that any fraction with denominator equal to 17 has a recurring decimal expansion. Extend the argument to deal with all rational numbers.

1.2 Surds

Square roots, or surds, were the first examples of irrational numbers to be identified.

For Interest

Irrational numbers were considered *(ab)surd*.

It is important to work with surds without using a calculator. Except to a limited extent (the most common calculators can work with surds $\sqrt{n}$, provided that the integer n is not too big), calculators can only handle the decimal expansion of a surd, and then only to 9 or so decimal places. Using a calculator inevitably means, therefore, that answers obtained will be inexact. They may be very accurate, but they will not be perfect. It is important to be able to work without reference to a calculator when possible. The main properties of surds are these:

Key Fact 1.1 Properties of Surds

- For any $x \geq 0$, the number $\sqrt{x}$ is the **non-negative (positive or zero)** square root of x.
- $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$ for any $x, y \geq 0$.
- $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ for any $x \geq 0$, $y > 0$.

The result for $\sqrt{xy}$ and $\sqrt{\frac{x}{y}}$ can be seen, because

$$\left(\sqrt{x}\times\sqrt{y}\right)\times\left(\sqrt{x}\times\sqrt{y}\right) \;=\; \left(\sqrt{x}\times\sqrt{x}\right)\times\left(\sqrt{y}\times\sqrt{y}\right) \;=\; x\times y \;=\; xy$$

and so $\sqrt{xy} \;=\; \sqrt{x}\times\sqrt{y}$. Moreover, since $\frac{x}{y}\times y \;=\; x$, we have

$$\sqrt{\frac{x}{y}}\times\sqrt{y} \;=\; \sqrt{x}$$

and hence $\sqrt{\frac{x}{y}} \;=\; \frac{\sqrt{x}}{\sqrt{y}}$.

These results can be used in a variety of ways to establish exact identities between surds.

Example 1.2.1. *Simplify the following expressions:*

a) $\sqrt{8}$, *b)* $\sqrt{75}$, *c)* $\sqrt{18}\times\sqrt{2}$, *d)* $\frac{\sqrt{27}}{\sqrt{3}}$,
e) $\sqrt{40}\times\sqrt{2}$, *f)* $\sqrt{28}+\sqrt{63}$, *g)* $\sqrt{5}\times\sqrt{10}$, *h)* $3\sqrt{2}\times 4\sqrt{7}$.

(a) $\sqrt{8} \;=\; \sqrt{4\times 2} \;=\; \sqrt{4}\times\sqrt{2} \;=\; 2\sqrt{2}$,

(b) $\sqrt{75} \;=\; \sqrt{25\times 3} \;=\; \sqrt{25}\times\sqrt{3} \;=\; 5\sqrt{3}$,

(c) $\sqrt{18}\times\sqrt{2} \;=\; \sqrt{18\times 2} \;=\; \sqrt{36} \;=\; 6$,

(d) $\frac{\sqrt{27}}{\sqrt{3}} \;=\; \sqrt{\frac{27}{3}} \;=\; \sqrt{9} \;=\; 3$,

(e) $\sqrt{40}\times\sqrt{2} \;=\; \sqrt{40\times 2} \;=\; \sqrt{16\times 5} \;=\; \sqrt{16}\times\sqrt{5} \;=\; 4\sqrt{5}$,

(f) $\sqrt{28}+\sqrt{63} \;=\; \sqrt{4}\times\sqrt{7}+\sqrt{9}\times\sqrt{7} \;=\; 2\sqrt{7}+3\sqrt{7} \;=\; 5\sqrt{7}$,

(g) $\sqrt{5}\times\sqrt{10} \;=\; \sqrt{5}\times\left(\sqrt{5}\times\sqrt{2}\right) = \left(\sqrt{5}\times\sqrt{5}\right)\times\sqrt{2} \;=\; 5\sqrt{2}$,

(h) $3\sqrt{2}\times 4\sqrt{7} \;=\; 12\sqrt{2\times 7} \;=\; 12\sqrt{14}$.

Surds can also be used to handle algebraic problems:

Example 1.2.2. *Simplify the following expressions:*

(a) $\sqrt{x^5y^2}$, *(b)* $\sqrt{x^3yz^2}\times\sqrt{xy^2}$, *(c)* $\frac{\sqrt{p^5q}}{\sqrt{p^2q^3}}$

(a) $\sqrt{x^5y^2} \;=\; \sqrt{x^4y^2\times x} \;=\; x^2y\sqrt{x}$.

(b) $\sqrt{x^3yz^2}\times\sqrt{xy^2} \;=\; \sqrt{x^3yz^2\times xy^2} \;=\; \sqrt{x^4y^3z^2} \;=\; x^2yz\sqrt{y}$.

(c) $\frac{\sqrt{p^5q}}{\sqrt{p^2q^3}} \;=\; \sqrt{\frac{p^5q}{p^2q^3}} \;=\; \sqrt{\frac{p^3}{q^2}} \;=\; \frac{\sqrt{p^3}}{\sqrt{q^2}} \;=\; \frac{p\sqrt{p}}{q}$.

Example 1.2.3. *Solve the simultaneous equations*

$$y \;=\; \sqrt{x} \qquad\qquad y^3 \;=\; 2x$$

We see that

$$\begin{aligned} 2x &= y^3 \;=\; \sqrt{x}\times\sqrt{x}\times\sqrt{x} \;=\; x\sqrt{x} \\ x\sqrt{x}-2x &= 0 \\ x(\sqrt{x}-2) &= 0 \end{aligned}$$

and hence either $x=0$ or $\sqrt{x}=2$, so either $x=0$ or $x=4$.

Similar rules can be applied to cube and higher roots.

Example 1.2.4. *Simplify the following expressions:*
(*a*) $\sqrt[3]{16}$, (*b*) $\sqrt[3]{12}\times\sqrt[3]{18}$, (*c*) $\sqrt[5]{1215}$.

(a) $\sqrt[3]{16} = \sqrt[3]{8\times 2} = \sqrt[3]{8}\times\sqrt[3]{2} = 2\sqrt[3]{2}$,

(b) $\sqrt[3]{12}\times\sqrt[3]{18} = \sqrt[3]{12\times 18} = \sqrt[3]{216} = 6$,

(c) $\sqrt[5]{1215} = \sqrt[5]{243\times 5} = \sqrt[5]{243}\times\sqrt[5]{5} = 3\sqrt[5]{5}$.

We frequently want to remove a surd from the denominator of a fraction. This is done either by cancelling the same surd in the numerator, or else by 'multiplying top and bottom' by a suitable expression. This process is called **rationalising the denominator**.

Key Fact 1.2 Rationalising the Denominator

- For any $x > 0$,
$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}}\times\frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}}{x}$$
and so
$$\frac{x}{\sqrt{x}} = \sqrt{x}$$
- For any $y \geq 0$,
$$\frac{1}{x+\sqrt{y}} = \frac{1}{x+\sqrt{y}}\times\frac{x-\sqrt{y}}{x-\sqrt{y}} = \frac{x-\sqrt{y}}{x^2-y}$$

Note the use of the 'Difference of Two Squares' technique to rationalise the denominator when the denominator was $x+\sqrt{y}$. Multiplying by $\frac{x-\sqrt{y}}{x-\sqrt{y}}$ does not change the value of the expression, because this fraction is equal to 1.

Example 1.2.5. *Write in simplified surd form:*
(*a*) $\frac{1}{\sqrt{2}}$, (*b*) $\frac{6}{\sqrt{2}}$, (*c*) $\frac{3\sqrt{2}}{\sqrt{10}}$, (*d*) $\frac{1}{3-\sqrt{2}}$

(a) $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$,

(b) $\frac{6}{\sqrt{2}} = \frac{3\times 2}{\sqrt{2}} = 3\sqrt{2}$,

(c) $\frac{3\sqrt{2}}{\sqrt{10}} = \frac{3\sqrt{2}}{\sqrt{5}\times\sqrt{2}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$,

(d) $\frac{1}{3-\sqrt{2}} = \frac{1}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{3+\sqrt{2}}{7}$

Using the 'Difference of Two Squares' method to rationalise the denominator, as shown in Example 1.2.5, is a surprisingly useful technique.

Example 1.2.6. *Find a positive integer n such that $\sqrt{n+1}-\sqrt{n} < 10^{-3}$.*

Note that
$$0 < \sqrt{n+1}-\sqrt{n} = \left(\sqrt{n+1}-\sqrt{n}\right)\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}\,.$$
Now $n+1 > n$, and so $\sqrt{n+1} > \sqrt{n}$, and hence $\sqrt{n+1}+\sqrt{n} > 2\sqrt{n}$. This tells us that
$$0 < \sqrt{n+1}-\sqrt{n} < \frac{1}{2\sqrt{n}}\,,$$
and we see that $\sqrt{n+1}-\sqrt{n} < 10^{-3}$ will be true if $2\sqrt{n} \geq 1000$, and so if $n \geq 500^2 (= 250000)$.

EXERCISE 1B

1. Simplify the following:
   a) $\sqrt{3}\times\sqrt{3}$ b) $\sqrt{8}\times\sqrt{2}$ c) $\sqrt{3}\times\sqrt{12}$ d) $2\sqrt{5}\times 3\sqrt{5}$
   e) $\left(2\sqrt{7}\right)^2$ f) $\sqrt[3]{x}\times\sqrt[3]{x^2y}$ g) $\sqrt[4]{125}\times\sqrt[4]{5}$ h) $\left(2\sqrt[4]{x}\right)^4$
2. Simplify the following (assuming that $x, y > 0$):
   a) $\sqrt{18}$ b) $\sqrt{45}$ c) $\sqrt{675}$ d) $\sqrt{x^3y^5}$
   e) $\sqrt{2000}$ f) $\sqrt[3]{250}$ g) $\sqrt[4]{32x^4y^4}$ h) $\sqrt{x^3+2x^2y+xy^2}$.
3. Simplify the following (assuming that $x, y > 0$):
   a) $\sqrt{8}+\sqrt{18}$ b) $\sqrt{20}-\sqrt{5}$ c) $2\sqrt{20}+3\sqrt{45}$
   d) $\sqrt{x^3}+\sqrt{xy^2}$ e) $\sqrt{99}+\sqrt{44}-\sqrt{11}$ f) $\sqrt{52}-\sqrt{13}$
   g) $\sqrt{4x^2+4xy+y^2}-\sqrt{x^2+2xy+y^2}$
4. Simplify the following:
   a) $\frac{\sqrt{8}}{\sqrt{2}}$ b) $\frac{\sqrt{40}}{\sqrt{20}}$ c) $\frac{\sqrt{3}}{\sqrt{48}}$ d) $\frac{\sqrt{50}}{\sqrt{200}}$
   e) $\frac{1}{\sqrt{5}}$ f) $\frac{3\sqrt{5}}{\sqrt{3}}$ g) $\frac{4\sqrt{2}}{\sqrt{12}}$ h) $\frac{2\sqrt{18}}{9\sqrt{12}}$
   i) $\frac{1}{2-\sqrt{3}}$ j) $\frac{1}{3\sqrt{5}-5}$ k) $\frac{4\sqrt{3}}{2\sqrt{6}+3\sqrt{2}}$ l) $\frac{12}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$
5. You are given that, correct to 12 decimal places, $\sqrt{26} = 5.099019513593$. Find the value of $\sqrt{650}$ correct to 10 decimal places.
6. Solve the simultaneous equations:
$$7x-(3\sqrt{5})y \;=\; 9\sqrt{5} \qquad (2\sqrt{5})x+y \;=\; 34$$
7. Assuming that $x > 0$, show that $\frac{\sqrt{x}}{\sqrt{x^2+x+x}} = \sqrt{x+1}-x\sqrt{x}$.

8*. Assuming that $x > 1$, evaluate
$$\frac{1}{\sqrt{x+\sqrt{x^2-x}}}-\sqrt{1-\sqrt{1-x^{-1}}}$$

9*. Put the following numbers in ascending order: $7-4\sqrt{3}$, $8-3\sqrt{7}$, $9-4\sqrt{5}$, $10-3\sqrt{11}$.

1.3 Indices

When mathematicians started solving quadratic, cubic and quartic equations, they wrote expressions like xx, xxx and $xxxx$ to denote the repeated product of a variable x with itself (just as abc is the product of a, b and c). It was found to be more economical to use the notations x^2, x^3 and x^4 instead, and so index notation was invented. However, index notation is not just a method of writing expressions conveniently; it introduces a new method of thought about number and algebra without which much of modern mathematics would be impossible.

1.3.1. POSITIVE INDICES

In general the symbol a^m stands for the result of multiplying m copies of a together:
$$a^m \;=\; \underbrace{a\times a\times\cdots\times a}_{m \text{ copies}}$$

This operation is described in words as 'a raised to the m^{th} power', or 'a to the power m' or even just 'a to the m'. The number a is called the **base**, and the number m the **index**. For the present, while a can be any number, m must be a positive integer. We shall extend consideration to negative indices below.
Expressions in index notation can be simplified, subject to a few simple rules:

Key Fact 1.3 Rules for Positive Indices

- $a^m \times a^n = a^{m+n}$,
- $a^m \div a^n = a^{m-n}$, if $m > n$,
- $(a^m)^n = a^{mn}$,
- $(ab)^m = a^m b^m$.

- $a^m \times a^n = \underbrace{a \times a \times \cdots \times a}_{m \text{ copies}} \times \underbrace{a \times a \times \cdots \times a}_{n \text{ copies}} = \underbrace{a \times a \times \cdots \times a}_{m+n \text{ copies}} = a^{m+n}$
- $a^m \div a^n = \underbrace{a \times a \times \cdots \times a}_{m \text{ copies}} \div \underbrace{a \times a \times \cdots \times a}_{n \text{ copies}} = \underbrace{a \times a \times \cdots \times a}_{m-n \text{ copies}} = a^{m-n}$
- $(a^m)^n = \underbrace{\underbrace{a \times a \times \cdots \times a}_{m \text{ copies}} \times \cdots \times \underbrace{a \times a \times \cdots \times a}_{m \text{ copies}}}_{n \text{ brackets}} = \underbrace{a \times a \times \cdots \times a}_{mn \text{ copies}} = a^{mn}$
- $(ab)^m = \underbrace{ab \times ab \times \cdots \times ab}_{m \text{ copies}} = \underbrace{a \times a \times \cdots \times a}_{m \text{ copies}} \times \underbrace{b \times b \times \cdots \times b}_{m \text{ copies}} = a^m \times b^m$

It is important to remember that, until we meet logarithms, the last of these rules is the only rule that can be applied to powers of different bases.

Example 1.3.1. *Simplify* $(2a^2b)^3 \div 4a^4b$.

Applying the rules,

$$(2a^2b)^3 \div 4a^4b \;=\; 2^3(a^2)^3b^3 \div 4a^4b \;=\; 8a^6b^3 \div 4a^4b \;=\; 2a^2b^2$$

For Interest

A common error is to write $2^3 \times 3^5 = 6^8$, multiplying the bases as well as adding the indices. Avoid it!

1.3.2. Zero and Negative Indices

The previous definition for a^m makes no sense if m is not a positive integer. Nevertheless it is possible to extend the definition of a^m to allow m to be any integer (provided that a is non-zero). If we look at the following table:

n	5	4	3	2	1
$\mathbf{2^n}$	32	16	8	4	2
$\mathbf{3^n}$	243	81	27	9	3

Every time the index n decreases by 1, the value of 2^n halves, and the value of 3^n is a third of its previous value. It is natural to extend the process

n	5	4	3	2	1	0	−1	−2	−3
$\mathbf{2^n}$	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$\mathbf{3^n}$	243	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

It seems that 2^0 and 3^0 should both be defined to be 1, while 2^{-m} should be the same as $\frac{1}{2^m}$, and 3^{-m} should be the same as $\frac{1}{3^m}$. This observation can be extended to any non-zero base a, and the resulting extension enables the previous rules for positive integer indices to be extended to general integer indices (and non-zero base).

Key Fact 1.4 Rules for Integer Indices

- $a^m \times a^n = a^{m+n}$,
- $a^m \div a^n = a^{m-n}$,
- $(a^m)^n = a^{mn}$,
- $(ab)^m = a^m b^m$,
- $a^0 = 1$,
- $a^{-m} = \frac{1}{a^m}$

To show that the rules for positive integer indices can be extended (for non-zero base) to all integer indices, we need to check a number of cases. Here are two of them (m and n are positive integers):

$$a^m \times a^{-n} = a^m \times \frac{1}{a^n} = \frac{1}{a^n \div a^m} = \frac{1}{a^{n-m}} = a^{m-n} \qquad (m < n),$$
$$(a^m)^{-n} = \frac{1}{(a^m)^n} = \frac{1}{a^{mn}} = a^{-mn}.$$

Example 1.3.2. *Simplify the following:*
(*a*) $4a^2b \times (3ab^{-1})^{-2}$, (*b*) $4^3 \times 2^{-5}$, (*c*) $(2xy^2z^3)^2 \div (2x^2y^3z)$

(a) $4a^2b \times (3ab^{-1})^{-2} = 4a^2b \times 3^{-2}a^{-2}b^2 = \frac{4}{9}b^3$,

(b) $4^3 \times 2^{-5} = (2^2)^3 \times 2^{-5} = 2^6 \times 2^{-5} = 2^1 = 2$,

(c) $(2xy^2z^3)^2 \div (2x^2y^3z) = 4x^2y^4z^6 \div 2x^2y^3z = 2yz^5$.

EXERCISE 1C

1. Simplify the following, writing each answer as a power of 2:
   a) $2^{11} \times (2^5)^3$ *b)* $(2^3)^2 \times (2^2)^3$ *c)* 4^3 *d)* 8^2
   e) $\frac{2^7 \times 2^8}{2^{13}}$ *f)* $\frac{2^2 \times 2^3}{(2^2)^2}$ *g)* $4^2 \div 2^4$ *h)* $2 \times 4^4 \div 8^3$
2. Simplify the following:
   a) $a^2 \times a^3 \times a^7$ *b)* $c^7 \div c^3$ *c)* $(e^5)^4$
   d) $5g^5 \times 3g^3$ *e)* $(2a^2)^3 \times (3a)^2$ *f)* $(4x^2y)^2 \times (2xy^3)^3$
   g) $(6ac^3)^2 \div (9a^2c^5)$ *h)* $(49r^3s^2)^2 \div (7rs)^3$ *i)* $(3h^2)^{-2}$
   j) $(\frac{1}{2}j^{-2})^{-3}$ *k)* $(3n^{-2})^4 \times (9n)^{-1}$ *l)* $(2q^{-2})^{-2} \div \left(\frac{4}{q}\right)^2$
3. Solve the following equations:
   a) $3^x = \frac{1}{9}$ *b)* $5^y = 1$ *c)* $2^z \times 2^{z-3} = 32$
   d) $7^{3x} \div 7^{x-2} = \frac{1}{49}$ *e)* $4^y \times 2^y = 8^{120}$ *f)* $3^t \times 9^{t+3} = 27^2$
4. Write $8^3 \times 4$ as a power of 2.
5. Simplify $\left(\frac{1}{\sqrt{3}}\right)^9$.
6. Solve the equation $\frac{3^{5x+2}}{9^{1-x}} = \frac{27^{4+3x}}{729}$.
7. Why do we not need brackets when considering powers of powers: in other words, why is a^{m^n} equal to $a^{(m^n)}$, and not to $(a^m)^n$?

8*. Which of 3^{4^3} and 4^{3^3} is bigger?

1.3.3. Fractional Indices

Up to now, we have assumed that the rules for indices work for integer indices. What can we deduce if we were to assume that the rules were still true for fractional indices? It would follow that

$$(a^{\frac{1}{2}})^2 = a^{\frac{1}{2}\times 2} = a^1 = a$$

for any positive a. Since $a^{\frac{1}{2}}$ squares to a, we deduce that $a^{\frac{1}{2}}$ is either $\sqrt{a}$ or $-\sqrt{a}$. Just as with surds, we **define** $a^{\frac{1}{2}}$ to be positive, so that $a^{\frac{1}{2}} = \sqrt{a}$. More generally we can show that $(a^{\frac{1}{m}})^m = a$, so that $a^{\frac{1}{m}} = \sqrt[m]{a}$ for any positive integer m and $a > 0$ (while it is possible to take m^{th} roots of negative numbers when m is odd, it is simpler to restrict fractional indices to strictly positive bases). Thus we can define the fractional power of any positive real number.

Key Fact 1.5 Rules for Fractional Indices

- $a^{\frac{1}{m}} = \sqrt[m]{a}$,
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$.

With these definitions, it should be noted that the rules for integer indices now hold for all fractional indices (and positive bases).

Example 1.3.3. *Simplify the following:*

(a) $16^{-\frac{3}{4}}$, (b) $(2\frac{1}{4})^{\frac{1}{2}}$, (c) $\dfrac{(2x^2y^2)^{-\frac{1}{2}}}{(2xy^{-2})^{\frac{3}{2}}}$

(a) $16^{-\frac{3}{4}} = (2^4)^{-\frac{3}{4}} = 2^{-3} = \frac{1}{8}$,

(b) $(2\frac{1}{4})^{\frac{1}{2}} = (\frac{9}{4})^{\frac{1}{2}} = \frac{3}{2}$,

(c) $\dfrac{(2x^2y^2)^{-\frac{1}{2}}}{(2xy^{-2})^{\frac{3}{2}}} = \dfrac{2^{-\frac{1}{2}}x^{-1}y^{-1}}{2^{\frac{3}{2}}x^{\frac{3}{2}}y^{-3}} = 2^{-2}x^{-\frac{5}{2}}y^2 = \dfrac{y^2}{4x^{\frac{5}{2}}}$

Exercise 1D

1. Evaluate the following:

   a) $25^{\frac{1}{2}}$ b) $36^{\frac{1}{2}}$ c) $81^{\frac{1}{4}}$ d) $16^{-\frac{1}{4}}$
   e) $1000^{-\frac{1}{3}}$ f) $27^{\frac{1}{3}}$ g) $64^{\frac{2}{3}}$ h) $125^{-\frac{4}{3}}$
   i) $4^{\frac{3}{2}}$ j) $27^{\frac{4}{3}}$ k) $32^{\frac{3}{5}}$ l) $4^{2\frac{1}{2}}$
   m) $10000^{-\frac{3}{4}}$ n) $(\frac{1}{125})^{-\frac{4}{3}}$ o) $(3\frac{3}{8})^{\frac{2}{3}}$ p) $(2.25)^{-\frac{1}{2}}$

2. Simplify the following expressions:

   a) $a^{\frac{1}{3}} \times a^{\frac{5}{3}}$ b) $3b^{\frac{1}{2}} \times 4b^{-\frac{3}{2}}$ c) $6c^{\frac{1}{4}} \times (4c)^{\frac{1}{2}}$
   d) $(d^2)^{\frac{1}{3}} \div (d^{\frac{1}{3}})^2$ e) $(2x^{\frac{1}{2}})^6 \times (\frac{1}{2}x^{\frac{3}{4}})^4$ f) $(24e)^{\frac{1}{3}} \div (3e)^{\frac{1}{3}}$
   g) $\dfrac{(5p^2q^4)^{\frac{1}{3}}}{(25pq^2)^{-\frac{1}{3}}}$ h) $(m^3n)^{\frac{1}{4}} \times (8mn^3)^{\frac{1}{3}}$ i) $\dfrac{(2x^2y^{-1})^{-\frac{1}{4}}}{(8x^{-1}y^2)^{-\frac{1}{2}}}$

3. Solve the following equations:

   a) $x^{\frac{1}{2}} = 8$ b) $x^{\frac{1}{3}} = 3$ c) $x^{\frac{2}{3}} = 4$
   d) $x^{\frac{3}{2}} = 27$ e) $x^{-\frac{3}{2}} = 8$ f) $x^{-\frac{2}{3}} = 9$
   g) $x^{\frac{3}{2}} = x\sqrt{2}$ h) $x^{\frac{3}{2}} = 2\sqrt{x}$ i) $4^x = 32$
   j) $9^y = \frac{1}{27}$ k) $16^z = 2$ l) $100^x = 1000$
   m) $8^z = \frac{1}{128}$ n) $(2^t)^3 \times 4^{t-1} = 16$ o) $\frac{9^y}{27^{2y+1}} = 81$

4*. Which is bigger, $2^{\frac{1}{2}}$ or $3^{\frac{1}{3}}$?

Chapter 1: Summary

- If $x \geq 0$, the square root of x, denoted $\sqrt{x}$, is the non-negative square root of x. Another word for a number which is the square root of another number is surd.
- For $x, y \geq 0$, $\sqrt{xy} = \sqrt{x}\sqrt{y}$.
- In fractions involving surds, the denominator may be rationalised as follows:

$$\frac{1}{\sqrt{x}} = \frac{1}{x}\sqrt{x} \qquad \frac{1}{x+\sqrt{y}} = \frac{x-\sqrt{y}}{x^2-y}$$

- The laws of indices state that

$$\begin{aligned} a^m \times a^n &= a^{m+n} & (a^m)^n &= a^{mn} \\ a^m \div a^n &= a^{m-n} & (ab)^m &= a^m b^m \\ a^0 &= 1 & a^{-n} &= \frac{1}{a^n} \end{aligned}$$

  These identities hold:

  ★ for all $a, b > 0$ and any values of m and n, or

  ★ for all non-zero a, b any integer values of m and n.

2

Coordinate Geometry

In this chapter we will learn to:

- find the length, gradient and midpoint of a line segment, given the coordinates of its end points,
- find the equation of a straight line, given sufficient information,
- understand and use the relationships between the gradients of parallel and perpendicular lines,
- interpret and use linear equations in context.

2.1 Introducing a Coordinate System

We have already studied a fair amount of geometry. We will have studied properties of parallel lines, similar and right-angled triangles, meeting Pythagoras' Theorem along the way. We are also aware of a number of results (mostly about angles) concerning figures constructed inside circles (like 'the angle at the centre is twice the angle at the circumference'), and we are aware of a number of results concerning the angles in quadrilaterals and polygons.

Armed with these results, and a good diagram of a problem, we can answer a pleasing number of questions. At the same time, we have already seen some benefits of the interplay between algebra and geometry. We have seen that straight lines can be represented by linear equations, and that the intersection of two lines can be determined by solving simultaneous equations. These results are the starting point for our considerations in this chapter. We will see that it is possible to implement a large number of geometric ideas and constructions solely in algebraic terms, and geometric results are obtained by solving suitable equations.

To be able to work this algebraic approach to geometry, we need to be able to describe points in space in terms of numbers. We do this by introducing a **coordinate system**. Before we start, we need to decide what sort of space we want to describe.

- Much mathematics is performed in a two-dimensional world; we can imagine an infinite sheet of flat paper, or an infinite whiteboard; this space is colloquially called **the plane**.
- Alternatively, we could be interested in a fully three-dimensional space, and want to be able to describe the position of anywhere in an infinite three-dimensional world.

For Interest

Mathematicians are not content with stopping at three dimensions, and the geometry of higher dimensional spaces can be investigated. We will not need to go that far!

Restricting our attention to the two-dimensional plane, we need a system whereby we can describe the position of any point on the plane. There are many ways of doing this, but we will confine our attention to the **Cartesian coordinate system**, discovered by the 17^{th} century French mathematician René Descartes (and also by Pierre de Fermat). To do so, we need to make a number of (basically arbitrary) choices:

- we need to identify one specific point in the plane, called the **origin**, and denoted by O,
- we need to choose two preferred directions, each perpendicular to the other,
- we need to choose a unit of length to be used for all measurements.

The position of a point P in the plane can be expressed by two numbers. It is possible to move from the origin O to the point P by moving a distance x parallel to the first direction, and then a distance y parallel to the second direction.
It is conventional to have the first preferred direction drawn horizontally on the page, and the second preferred direction drawn vertically. The distances x and y are signed; a negative value of x means that P is to the left of O, and a negative value of y means that P is below O. We call x and y the **coordinates** of P, and write them as a pair inside a set of brackets, (x, y).

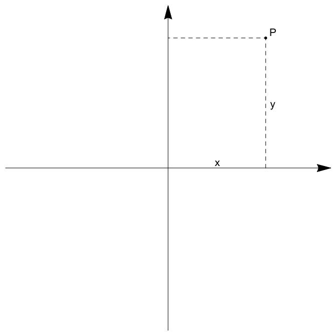


Figure 2.1

If we wanted to work three-dimensionally, a similar procedure would require us to define three mutually perpendicular directions as well as the origin, and we could describe the position of any point in this space by a triple of coordinates (x, y, z).

2.2 The Length of a Line Segment

Two points A and B have coordinates $(-2, 1)$ and $(4, 4)$ respectively. The straight line joining A to B is called the **line segment** AB. The length of the line segment AB is the distance between the points A and B.
A third point C is created by drawing lines from A and B parallel to the coordinate axes. Note that ABC is a right-angled triangle. It is clear that C has the same x-coordinate as B, namely 4, and the same y-coordinate as A, namely 1, so that C has coordinates $(4, 1)$. It is clear that AC has length $4 - (-2) = 6$, while BC has length $4 - 1 = 3$. Using Pythagoras' Theorem, we deduce that the length of $AB = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$.

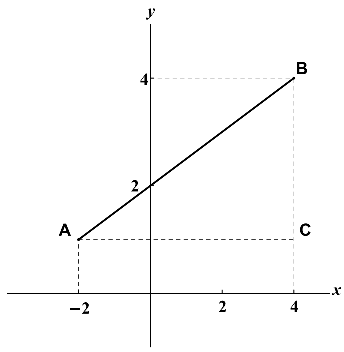


Figure 2.2

While we could use a calculator to write this length to a given accuracy, it is better to be exact and write the length as a surd.

The important thing to notice is that this method can be used wherever A and B are in the plane, and Pythagoras' Theorem will give us a formula to use for the length of AB, even though the diagram we might have to draw looks different every time.
If the points A and B have coordinates (x_1, y_1) and (x_2, y_2) respectively, then the horizontal and vertical sides of the right-angled triangle with AB as hypotenuse have lengths $|x_1 - x_2|$, $|y_1 - y_2|$ respectively. We can therefore use Pythagoras' Theorem, and obtain a general formula for the distance between two points on the plane:

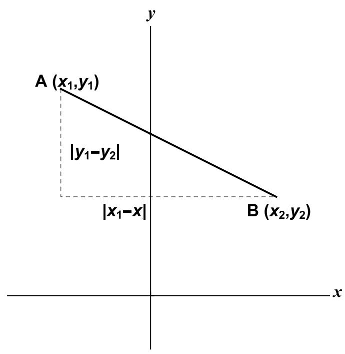


Figure 2.3

Key Fact 2.1 The Length of a Line Segment

If A and B have coordinates (x_1, y_1) and (x_2, y_2), then the distance between A and B is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The key point to remember is that we can use this formula without drawing any diagram. The distance between two points can be found by purely algebraic manipulations.

Example 2.2.1. *Show that the triangle ABC is isosceles, where A, B and C have coordinates $(1,3)$, $(6,-1)$ and $(5,8)$ respectively.*

We see that $AB = \sqrt{(1-6)^2 + (3--1)^2} = \sqrt{5^2 + 4^2} = \sqrt{41}$, while $AC = \sqrt{(1-5)^2 + (3-8)^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$, and so $AB = AC$.

Example 2.2.2. *The triangle PQR is right-angled, where P, Q and R have coordinates $(1,4)$, $(3,1)$ and $(k,8)$. Find the possible values of k.*

We calculate $PQ = \sqrt{2^2 + 3^2} = \sqrt{13}$, $PR = \sqrt{(k-1)^2 + 4^2} = \sqrt{k^2 - 2k + 17}$, and $QR = \sqrt{(k-3)^2 + 7^2} = \sqrt{k^2 - 6k + 58}$. There are three cases to consider, depending on where the right angle is.

- If the right angle is at P, then

$$\begin{aligned} PQ^2 + PR^2 &= QR^2 \\ 13 + k^2 - 2k + 17 &= k^2 - 6k + 58 \\ 4k &= 28 \\ k &= 7. \end{aligned}$$

- If the right angle is at Q, then

$$\begin{aligned} PQ^2 + QR^2 &= PR^2 \\ 13 + k^2 - 6k + 58 &= k^2 - 2k + 17 \\ 54 &= 4k \\ k &= \tfrac{27}{2}. \end{aligned}$$

- If the right angle is at R, then

$$\begin{aligned} PR^2 + QR^2 &= PQ^2 \\ k^2 - 2k + 17 + k^2 - 6k + 58 &= 13 \\ 2k^2 - 8k + 62 &= 0 \\ k^2 - 4k + 31 &= 0 \end{aligned}$$

  Since this quadratic has discriminant $-100 < 0$, there are no solutions for k.

Thus the only possible values for k are 7 and $\frac{27}{2}$.

2.3 The Midpoint of a Line Segment

Consider a line segment AB, where the endpoints A and B have coordinates (x_1, y_1) and (x_2, y_2) respectively. Let M be the midpoint of the line segment AB, and construct the points C and D by drawing lines parallel to the coordinate axes as shown.
It is clear that C has coordinates (x_1, y_2). Moreover the triangles ACB and MDB are similar, and hence the lengths CD and DB are equal. Thus D is the midpoint of CB, and hence it is clear that D has coordinates $\left(\frac{1}{2}(x_1 + x_2), y_2\right)$. Moreover AC is twice the length of MD, and hence the y-coordinate of D is $y_2 + \frac{1}{2}(y_1 - y_2) = \frac{1}{2}(y_1 + y_2)$.

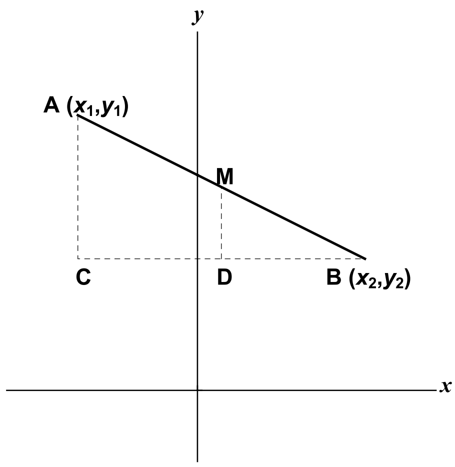


Figure 2.4

Key Fact 2.2 The Midpoint of a Line Segment

If A and B have coordinates (x_1, y_1) and (x_2, y_2), then the midpoint of AB has coordinates

$$\left(\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2)\right)$$

Where coordinate geometry comes into its own is when it is coupled with standard geometry, and we use our standard geometrical knowledge and insight to understand what calculations to perform.

Example 2.3.1. *The parallelogram $ABCD$ has coordinates $A(1,4)$, $B(5,7)$ and $D(3,5)$. What are the coordinates of C?*

We know that the diagonals of a parallelogram bisect each other. The midpoint of BD has coordinates $(4,6)$, and is also the midpoint of AC. Thus C must have coordinates (u, v), where $\left(\frac{1}{2}(1+u), \frac{1}{2}(4+v)\right) = (4,6)$, and hence $u = 7$ and $v = 8$. The coordinates of C are $(7,8)$.

2.4 The Gradient of a Line Segment

The gradient of a line is its slope. The steeper the line, the larger the gradient. Unlike coordinates, the gradient of a line is a property of the whole line, and not of just one point on it. The gradient is colloquially defined as the ratio

$$\frac{\text{RISE}}{\text{RUN}}$$

where the RISE is the amount that the line has increased by between two points on that line, and the RUN is the amount that has been travelled from left to right between those two points. Although the rise and the run will differ, depending on the two points on the line that are chosen, their ratio does not.

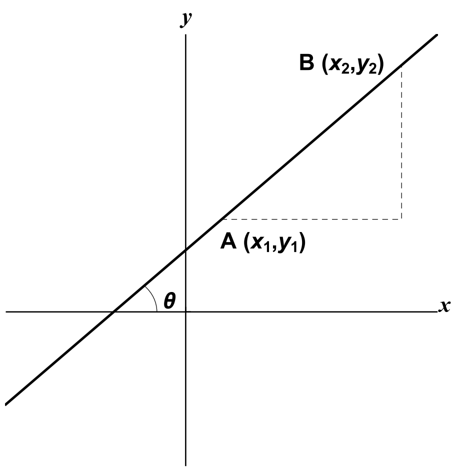


Figure 2.5

Again, the convenience of coordinate geometry is that this calculation can be performed very neatly in coordinates, without the need to draw a diagram. If A has coordinates (x_1, y_1), and B has coordinates (x_2, y_2), then the line segment has risen by $y_2 - y_1$ between A and B, and hence the rise is $y_2 - y_1$. Similarly, the run between these two points is $x_2 - x_1$.

Key Fact 2.3 The Gradient of a Line Segment

The gradient of the line segment AB, where A and B have coordinates (x_1, y_1) and (x_2, y_2), is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

The gradient is the tangent $\tan\theta$ of the angle that the line segment makes with the x-axis.

This formula works whether the coordinates of A and B are positive or negative.

Example 2.4.1. *Find the gradient of the line segment AB, where A and B have coordinates:*

a) $A(3,5)$, $B(7,12)$ *b)* $A(-4,5)$, $B(6,-4)$

The gradient in the first case is $\frac{12-5}{7-3} = \frac{7}{4}$. In the second case it is $\frac{-4-5}{6-(-4)} = -\frac{9}{10}$.

For Interest

The formula fails to work if $x_1 = x_2$, because we are trying to divide by 0. In this case, the two points have the same x-coordinate, and the line segment is vertical. In this case, we say that the gradient of the line is infinite, written ∞.

Even more importantly, we do not have to know that A is 'to the left of' B. In other words, we do not need to know that the rise is positive. Since

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

the formula for the gradient tells us that the gradient of AB is the same as the gradient of BA. They are, after all, the same line. This is particularly important when we are performing algebraic calculations, and we might not know (at least initially) whether $x_1 < x_2$ or the reverse.

Example 2.4.2. *The lines AB and CD have the same gradient, where $A(3,1)$, $B(p,p)$, $C(5,3)$ and $D(2p,2p)$. What is the value of p?*

Since the two gradients are the same, we see that

$$\begin{aligned}
\frac{p-1}{p-3} &= \frac{2p-3}{2p-5} \\
(p-1)(2p-5) &= (p-3)(2p-3) \\
2p^2 - 7p + 5 &= 2p^2 - 9p + 9 \\
2p &= 4
\end{aligned}$$

and hence we deduce that $p = 2$.

Recall that lines with the same gradient are called **parallel**.

Example 2.4.3. *Prove that the points $A(1,1)$, $B(5,3)$, $C(3,0)$ and $D(-1,-2)$ form a parallelogram.*

There are two ways in which this can be done. We can calculate the lengths of the four sides:

$$\begin{aligned} AB &= \sqrt{(5-1)^2+(3-1)^2} = \sqrt{20} \\ CD &= \sqrt{(-1-3)^2+(-2-0)^2} = \sqrt{20} \\ BC &= \sqrt{(3-5)^2+(0-3)^2} = \sqrt{13} \\ AD &= \sqrt{(-1-1)^2+(-2-1)^2} = \sqrt{13} \end{aligned}$$

Since opposite pairs of sides have the same length, we have a parallelogram.

Alternatively, we could calculate the gradients of the sides. The sides AB, BC, CD and DA have respective gradients

$$\tfrac{3-1}{5-1} = \tfrac{1}{2} \qquad \tfrac{0-3}{3-5} = \tfrac{3}{2} \qquad \tfrac{-2-0}{-1-3} = \tfrac{1}{2} \qquad \tfrac{1-(-2)}{1-(-1)} = \tfrac{3}{2}$$

Since opposite pairs of sides are parallel, $ABCD$ is a parallelogram.

EXERCISE 2A

Do not use a calculator. Where appropriate, leave square roots in your answers.

1. Find the lengths of the line segments joining these pairs of points. Where necessary assume that $a > 0$ and $p > q > 0$.
   - *a)* $(2,5)$ and $(7,17)$
   - *b)* $(-3,2)$ and $(1,-1)$
   - *c)* $(4,-5)$ and $(-1,0)$
   - *d)* $(-3,-3)$ and $(-7,3)$
   - *e)* $(2a,a)$ and $(10a,-14a)$
   - *f)* $(a+1,2a+3)$ and $(a-1,2a-1)$
   - *g)* $(2,9)$ and $(2,-14)$
   - *h)* $(12a,5b)$ and $(3a,5b)$
   - *i)* (p,q) and (q,p)
   - *j)* $(p+4q,p-q)$ and $(p-3q,p)$
2. Show that the points $(1,-2)$, $(6,-1)$, $(9,3)$ and $(4,2)$ are vertices of a parallelogram.
3. Show that the triangle formed by the points $(-3,-2)$, $(2,-7)$ and $(-2,5)$ is isosceles.
4. Show that the points $(7,12)$, $(-3,-12)$ and $(14,-5)$ lie on a circle with centre $(2,0)$.
5. Find the coordinates of the midpoints of the line segments joining these pairs of points.
   - *a)* $(2,11)$, $(6,15)$
   - *b)* $(5,7)$, $(-3,9)$
   - *c)* $(-2,-3)$, $(1,-6)$
   - *d)* $(-3,4)$, $(-8,5)$
   - *e)* $(p+2,3p-1)$, $(3p+4,p-5)$
   - *f)* $(p+3,q-7)$, $(p+5,3-q)$
   - *g)* $(p+2q,2p+13q)$, $(5p-2q,-2p-7q)$
   - *h)* $(a+3,b-5)$, $(a+3,b+7)$
6. $A(-2,1)$ and $B(6,5)$ are the opposite ends of the diameter of a circle. Find the coordinates of its centre.
7. $M(5,7)$ is the midpoint of the line segment joining $A(3,4)$ to B. Find the coordinates of B.
8. $A(1,-2)$, $B(6,-1)$, $C(9,3)$ and $D(4,2)$ are the vertices of a parallelogram. Verify that the midpoints of the diagonals AC and BD coincide.
9. Which one of the points $A(5,2)$, $B(6,-3)$ and $C(4,7)$ is the midpoint of the other two? Check your answer by calculating two distances.
10. Find the gradients of the lines joining the following pairs of points.
    - *a)* $(3,8)$, $(5,12)$
    - *b)* $(1,-3)$, $(-2,6)$
    - *c)* $(-4,-3)$, $(0,-1)$
    - *d)* $(-5,-3)$, $(3,-9)$
    - *e)* $(p+3,p-3)$, $(2p+4,-p-5)$
    - *f)* $(p+3,q-5)$, $(q-5,p+3)$
    - *g)* $(p+q-1,q+p-3)$, $(p-q+1,q-p+3)$
    - *h)* $(7,p)$, $(11,p)$
11. Find the gradients of the lines AB and BC where A is $(3,4)$, B is $(7,6)$ and C is $(-3,1)$. What can you deduce about the points A, B and C?

12. The point $P(x,y)$ lies on the straight line joining $A(3,0)$ and $B(5,6)$. Find expressions for the gradients of AP and PB. Hence show that $y = 3x - 9$.

13. A line joining a vertex of a triangle to the midpoint of the opposite side is called a median. Find the length of the median AM in the triangle $A(-1,1)$, $B(0,3)$, $C(4,7)$.

14. A triangle has vertices $A(a,b)$, $B(p,q)$ and $C(u,v)$.

    a) Find the coordinates of M, the midpoint of AB, and N, the midpoint of AC.

    b) Show that MN is parallel to BC.

15. The points $A(2,1)$, $B(2,7)$ and $C(-4,-1)$ form a triangle. M is the midpoint of AB and N is the midpoint of AC.

    a) Find the lengths of MN and BC.   *b)* Show that $BC = 2MN$.

16. The vertices of a quadrilateral $ABCD$ are $A(1,1)$, $B(7,3)$, $C(9,-7)$ and $D(-3,-3)$. The points P, Q, R and S are the midpoints of AB, BC, CD and DA respectively.

    a) Find the gradient of each side of $PQRS$.   *b)* What type of quadrilateral is $PQRS$?

17. The origin O and the points $P(4,1)$, $Q(5,5)$ and $R(1,4)$ form a quadrilateral.

    a) Show that OR is parallel to PQ.   *b)* Show that OP is parallel to RQ.
    c) Show that $OP = OR$.   *d)* What shape is $OPQR$?

18. The origin O and the points L(-2, 3), M(4, 7) and N(6, 4) form a quadrilateral.

    a) Show that $ON = LM$.   *b)* Show that ON is parallel to LM.
    c) Show that $OM = LN$.   *d)* What shape is $OLMN$?

19. The vertices of a quadrilateral $PQRS$ are $P(1,2)$, $Q(7,0)$, $R(6,-4)$ and $S(-3,-1)$.

    a) Find the gradient of each side of the quadrilateral.

    b) What type of quadrilateral is $PQRS$?

20. The vertices of a quadrilateral are $T(3,2)$, $U(2,5)$, $V(8,7)$ and $W(6,1)$. The midpoints of UV and VW are M and N respectively. Show that the triangle TMN is isosceles.

21. The vertices of a quadrilateral $DEFG$ are $D(3,-2)$, $E(0,-3)$, $F(-2,3)$ and $G(4,1)$.

    a) Find the length of each side of the quadrilateral.

    b) What type of quadrilateral is $DEFG$?

22. The points $A(2,3)$, $B(4,x)$ and $C = (2x,-3)$ are such that $BC = AC$. What are the possible values of x?

23*. The triangle ABC has vertices $A(a,b)$, $B(p,q)$ and $C(u,v)$. The points L, M and N are the midpoints of BC, AC and AB respectively.

    a) Write down the coordinates of L, M and N.

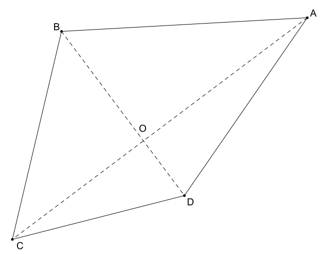
    b) If G is the point $\left(\frac{1}{3}(a+p+u), \frac{1}{3}(b+q+v)\right)$, show that G lies on the line AL.

    c) Explain why the three medians of the triangle all pass through one point (we say that the medians are **concurrent**).

24*. The quadrilateral $ABCD$ has perpendicular diagonals. Let O be the point of intersection of the two diagonals. Suppose that $OA = a$, $OB = b$, $OC = c$, and $OD = d$.

    a) Write down expressions for AB and CD

    b) Prove that $AB^2 + CD^2 = AD^2 + BC^2$.



25*. We know that the diagonals of a parallelogram bisect each other. Prove now that any quadrilateral $ABCD$ whose diagonals bisect each other is a parallelogram (*Hint: Define coordinate axes so that AC lies on the x-axis, with O being the midpoint of the diagonal.*).

2.5 The Equation of a Line

There are various ways by which we can specify a line. We might, for instance, know the line's gradient and one of the points it passes through. Suppose then that a straight line has gradient m, and crosses the y-axis at the point $(0, c)$. In this case, c is called the y**-intercept**. If we consider a point (x, y) on the line, then we calculate the line's gradient to be

$$m = \frac{y-c}{x-0}$$

and hence we deduce that $y = mx + c$.

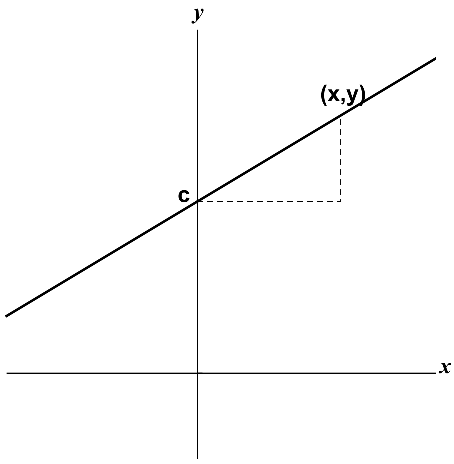


Figure 2.6

Key Fact 2.4 The Equation of a Line 1

The equation of the line with gradient m with y-intercept c is

$$y = mx + c$$

More generally, we might know the gradient of the line, but instead of knowing the y-intercept, we might have the coordinates of a point through which the line passes (to know the y-intercept is, of course, to know the point on the y-axis through which the line passes). Suppose that a straight line has gradient m, and passes through the point (x_1, y_1). If (x, y) is any point on the line, we calculate the gradient of the line to be

$$m = \frac{y - y_1}{x - x_1}$$

so the line has equation $y - y_1 = m(x - x_1)$.

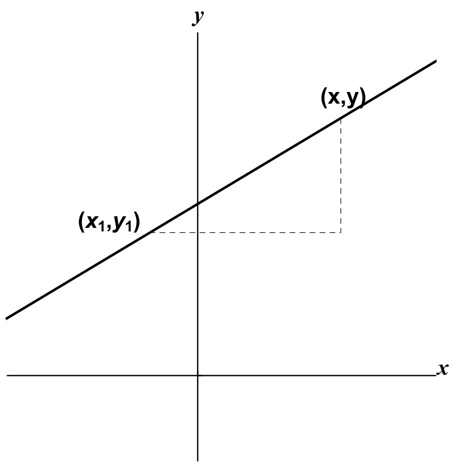


Figure 2.7

Key Fact 2.5 The Equation of a Line 2

The equation of the line with gradient m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

This formula clearly sets up the equation of a line with gradient m. Since both sides of the equation are equal to 0 when $x = x_1$ and $y = y_1$, it is clear that this line passes through the point (x_1, y_1). Note that a line with y-intercept c is a line passing through $(0, c)$, and so the $y = mx + c$ formula is a special case of this one.

What can we do if we do not know the gradient of the line? All that it takes to define a straight line is to know two points on that line. Suppose now that a line passes through the points (x_1, y_1) and (x_2, y_2). Then this line has gradient

$$\frac{y_2 - y_1}{x_2 - x_1}$$

and hence its equation is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

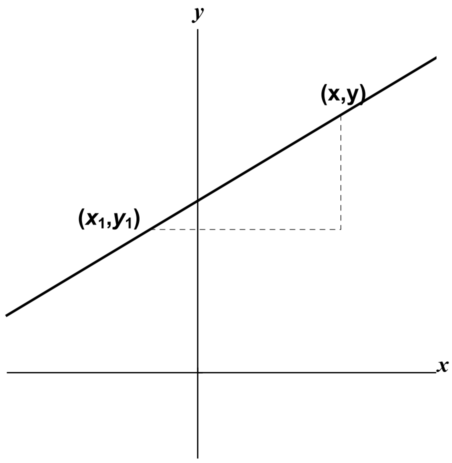


Figure 2.8

For Interest

When drawing straight line graphs in previous years, you were probably taught to plot three points before drawing the line. Three points were not necessary to define the line, but the third point was useful as a means of detecting calculation error in determining the coordinates of those points, or identifying if one of those points had been plotted incorrectly.

Key Fact 2.6 The Equation of a Line 3

The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

It is worth noting that both sides of this equation are equal to 0 at the point (x_1, y_1), and both sides are equal to 1 at (x_2, y_2). This is a useful point to remember, since it helps us to substitute the right values into the formulae correctly.

Example 2.5.1. *Find the equation of the line with gradient* -1 *passing through the point* $(-2, 3)$.

The equation is $y - 3 = -1\big(x - (-2)\big) = -x - 2$, or $y = 1 - x$.

Example 2.5.2. *Find the equation of the line passing through the points* $(3, 4)$ *and* $(-1, 2)$.

There are two possible approaches here. We could start by finding the gradient $\frac{4-2}{3-(-1)} = \frac{1}{2}$, so that the equation is $y - 4 = \frac{1}{2}(x - 3)$, or $y = \frac{1}{2}x + \frac{5}{2}$. Alternatively we could use the last formula, so that the equation is

$$\tfrac{1}{2}(y - 2) = \frac{y - 2}{4 - 2} = \frac{x - (-1)}{3 - (-1)} = \tfrac{1}{4}(x + 1)$$

which gives $y = \frac{1}{2}x + \frac{5}{2}$

We note that none of these methods for obtaining the equations of lines work if the line is 'vertical' (has infinite gradient). Lines of this type have equations of the form $x = k$ for some constant k. This means that the first two formulae fail to work, because we have no value for the gradient m that we can use, and the third formula fails because two points on the line will have the same x-coordinate, and hence the formula asks us to divide by 0. There should be little difficulty in recognising such lines, and finding their equations without reference to any of the three formulae above.

2.6 The equation $ax + by + c = 0$

It is untidy that we do not currently have a method of writing equations of lines which can represent **all** lines: the $y = mx + c$ shape is not good enough, since it does not handle vertical lines.

It is easy to see that an equation of the form $y = mx + c$ can be written as $mx - y + c = 0$, while a vertical line, with equation $x = k$, can be written in the form $x - k = 0$.

Food For Thought 2.1

For any straight line, constants a, b and c can be found such that the equation of the line is

$$ax + by + c = 0$$

Example 2.6.1. *Find the equation of the line passing through the points $(2,4)$ and $(5,-3)$ in the form $ax + by + c = 0$, where a, b and c are integers.*

Using the third formula, the equation is

$$\begin{aligned} \frac{y-4}{-3-4} &= \frac{x-2}{5-2} \\ -\tfrac{1}{7}(y-4) &= \tfrac{1}{3}(x-2) \\ -3(y-4) &= 7(x-2) \\ 7x + 3y - 26 &= 0 \end{aligned}$$

Example 2.6.2. *What is the equation of the line passing through A and the midpoint of BC, where A, B and C have coordinates $(1,4)$, $(-2,7)$ and $(4,11)$? Write the equation in the form $ax+by+c = 0$.*

The midpoint of BC has coordinates $(1,9)$. The desired line is a vertical one, with equation $x = 1$, or $x - 1 = 0$.

Example 2.6.3. *What is the gradient of the line $3x + 11y - 6 = 0$?*

This equation can be written as $y = -\frac{3}{11}x + \frac{6}{11}$, and hence its gradient is $-\frac{3}{11}$.

The last example shows another benefit of this new way of writing equations of lines. The $y = mx + c$ formulation might involve complicated fractions, whereas the new format only needs integers to describe the line.

Example 2.6.4. *One side of a parallelogram lies on the straight line with equation $3x-4y-7 = 0$. The point $(2,3)$ is a vertex of the parallelogram. Find the equation of one other side.*

The line $3x - 4y - 7 = 0$ can be written $y = \frac{3}{4}x - \frac{7}{4}$, and hence has gradient $\frac{3}{4}$. The point $(2,3)$ does not lie on this line. Hence another side of the parallelogram (the only other side we can be certain about) passes through $(2,3)$ with gradient $\frac{3}{4}$, and so has equation $y - 3 = \frac{3}{4}(x - 2)$, or $3x - 4y + 6 = 0$.

What this method of describing lines gains in elegance and convenience, however, it loses in uniqueness. There is no longer exactly one equation of a line. The equations

$$x + 3y - 1 = 0 \qquad 3x + 9y - 3 = 0 \qquad -2x - 6y + 2 = 0$$

all describe the same equation (each can be obtained from the other by multiplying through by some constant). However, the benefits of being able to describe all lines in a single manner outweigh this disadvantage.

2.7 The Point of Intersection of Two Lines

Where do the two lines $2x - y = 4$ and $3x + 2y + 1 = 0$ meet? How do we find the coordinates of the point of intersection of these two lines.
We want the point (x, y) that lies on both lines, and hence the values (x, y) must satisfy both equations. To find these values, then, we need to solve the two equations simultaneously. These particular equations have simultaneous solution $x = 1$, $y = -2$, so the point of intersection is $(1, -2)$.

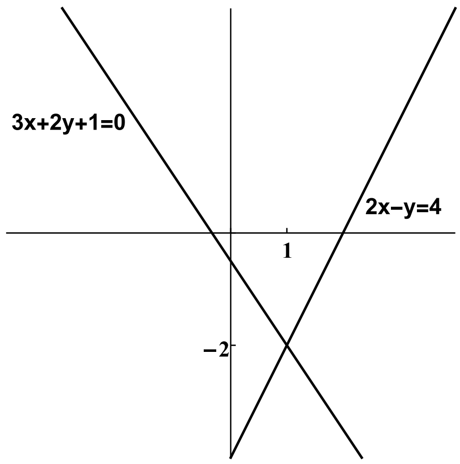


Figure 2.9

This technique will find the point of intersection of any pair of nonparallel lines. Solving simultaneous equations will also enable us to find the points of intersection of more complicated curves.

Exercise 2B

1. Test whether the given point lies on the straight line (or curve) with the given equation.
   a) $(1, 2)$ on $y = 5x - 3$,
   b) $(3, -2)$ on $y = 3x - 7$,
   c) $(3, -4)$ on $x^2 + y^2 = 25$,
   d) $(2, 2)$ on $3x^2 + y^2 = 40$,
   e) $\left(1, 1\frac{1}{2}\right)$ on $y = \frac{x+2}{3x-1}$,
   f) $\left(5p, \frac{5}{p}\right)$ on $y = \frac{5}{x}$,
   g) $(p, (p-1)^2 + 1)$ on $y = x^2 - 2x + 2$,
   h) $(t^2, 2t)$ on $y^2 = 4x$.

2. Find the equations of the straight lines through the given points with the gradients shown. Your final answers should not contain any fractions.
   a) $(2, 3)$, gradient 5,
   b) $(1, -2)$, gradient -3,
   c) $(0, 4)$, gradient $\frac{1}{2}$,
   d) $(-2, 1)$, gradient $-\frac{3}{8}$,
   e) $(0, 0)$, gradient -3,
   f) $(3, 8)$, gradient 0,
   g) $(-5, -1)$, gradient $-\frac{3}{4}$,
   h) $(-3, 0)$, gradient $\frac{1}{2}$,
   i) $(-3, -1)$, gradient $\frac{3}{8}$,
   j) $(3, 4)$, gradient $-\frac{1}{2}$,
   k) $(2, -1)$, gradient -2,
   l) $(-2, -5)$, gradient 3,
   m) $(0, -4)$, gradient 7,
   n) $(0, 2)$, gradient -1,
   o) $(3, -2)$, gradient $-\frac{5}{8}$,
   p) $(3, 0)$, gradient $-\frac{3}{5}$,
   q) $(d, 0)$, gradient 7,
   r) $(0, 4)$, gradient m,
   s) $(0, c)$, gradient 3,
   t) $(c, 0)$, gradient m.

3. Find the equations of the lines joining the following pairs of points. Leave your final answer without fractions and in one of the forms $y = mx + c$ or $ax + by + c = 0$.
   a) $(1, 4)$ and $(3, 10)$,
   b) $(4, 5)$ and $(-2, -7)$,
   c) $(3, 2)$ and $(0, 4)$,
   d) $(3, 7)$ and $(3, 12)$,
   e) $(10, -3)$ and $(-5, -12)$,
   f) $(3, -1)$ and $(-4, 20)$,
   g) $(2, -3)$ and $(11, -3)$,
   h) $(2, 0)$ and $(5, -1)$,
   i) $(-4, 2)$ and $(-1, -3)$,
   j) $(-2, -1)$ and $(5, -3)$,
   k) $(-3, 4)$ and $(-3, 9)$,
   l) $(-1, 0)$ and $(0, -1)$,
   m) $(2, 7)$ and $(3, 10)$,
   n) $(-5, 4)$ and $(-2, -1)$,
   o) $(0, 0)$ and $(5, -3)$,
   p) $(0, 0)$ and (p, q),
   q) (p, q) and $(p + 3, q - 1)$,
   r) $(p, -q)$ and (p, q),
   s) (p, q) and $(p + 2, q + 2)$,
   t) $(p, 0)$ and $(0, q)$.

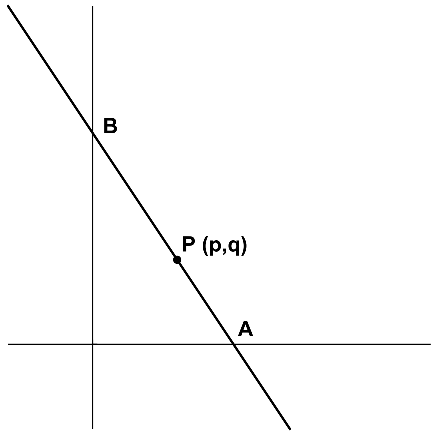
4. Find the gradients of the following lines.

a) $2x+y=7$,	b) $3x-4y=8$,	c) $5x+2y=-3$,
d) $y=5$,	e) $3x-2y=-4$,	f) $5x=7$,
g) $x+y=-3$,	h) $y=3(x+4)$,	i) $7-x=2y$,
j) $3(y-4)=7x$,	k) $y=m(x-d)$,	l) $px+qy=pq$.

5. Find the equation of the line through $(-2,1)$ parallel to $y=\frac{1}{2}x-3$.
6. Find the equation of the line through $(4,-3)$ parallel to $y+2x=7$.
7. Find the equation of the line through $(1,2)$ parallel to the line joining $(3,-1)$ and $(-5,2)$.
8. Find the equation of the line through $(3,9)$ parallel to the line joining $(-3,2)$ and $(2,-3)$.
9. Find the equation of the line through $(1,7)$ parallel to the x-axis.
10. Find the equation of the line through $(d,0)$ parallel to $y=mx+c$.
11. Find the points of intersection of the following pairs of straight lines.

a) $3x+4y=33$, $2y=x-1$,	b) $y=3x+1$, $y=4x-1$,
c) $2y=7x$, $3x-2y=1$,	d) $y=3x+8$, $y=-2x-7$,
e) $x+5y=22$, $3x+2y=14$,	f) $2x+7y=47$, $5x+4y=50$,
g) $2x+3y=7$, $6x+9y=11$,	h) $3x+y=5$, $x+3y=-1$,
i) $y=2x+3$, $4x-2y=-6$,	j) $ax+by=c$, $y=2ax$,
k) $y=mx+c$, $y=-mx+d$,	l) $ax-by=1$, $y=x$,

12. $ABCD$ is a rectangle, where A, B and C have coordinates $(0,0)$, $(6,0)$ and $(6,3)$ respectively. Let P be the midpoint of AB, and let Q be the midpoint of CD. The lines DP and QB meet the diagonal AC at the points M and N respectively. Show that $AM=MN=NC$.
13. Let P, with coordinates (p,q), be a fixed point on the line with equation $y=mx+c$, and let Q, with coordinates (r,s), be any other point on that line. Show that the gradient of PQ is m for all positions of Q.
14. There are some values of a, b and c for which the equation $ax+by+c=0$ does not represent a straight line. What are they?
15*. The point P has coordinates (p,q). The line ℓ passes through P, and has negative gradient $-m$, where $m>0$. The line ℓ meets the x-axis and y-axis at A and B respectively. Find an expression for the area of the triangle OAB. What happens when $q=-mp$?



16*. The points A and B have coordinates $(-a,0)$ and $(a,0)$ respectively. The line ℓ_1 passes through A with gradient m, while the line ℓ_2 passes through B with gradient $-m^{-1}$. Find the coordinates (u,v) of the point of intersection C of the lines ℓ_1 and ℓ_2, and show that $u^2+v^2=a^2$.
17*. The triangle ABC has coordinates $A(a_1,a_2)$, $B(b_1,b_2)$ and $C(c_1,c_2)$. A **median** of a triangle is the line passing through one vertex and the midpoint of the opposite edge (so that median through A passes through A and the midpoint of BC).
    a) Write down the equation of the median through A.
    b) Show that the point G with coordinates $\left(\frac{1}{3}(a_1+b_1+c_1), \frac{1}{3}(a_2+b_2+c_2)\right)$ lies on this median.
    c) Why does this show that all three medians of a triangle meet at a point (such lines are called **concurrent**)? The point where all three medians meet is called the **centroid**, or **centre of gravity**, of the triangle.

2.8 Perpendicular Lines

If two lines are parallel, then they have the same gradient. What can we say about the gradients of two lines which are perpendicular?

Certainly, if a line has a positive gradient, then the perpendicular line has a negative gradient, and vice versa. But we can be more precise than this.

If we have a line with positive gradient m, then we can pick points P and B on the line so that the run from P to B is 1, and so the rise from P to B is m. We have the 'gradient triangle' PAB.

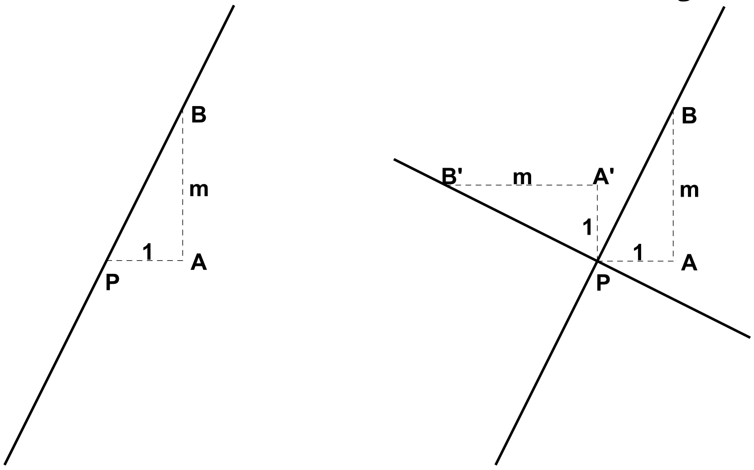


Figure 2.10

If we consider a second line which is perpendicular to the first, and which meets the first line at P, then this line can be obtained from the first by rotating it through $90°$ about the point P. The gradient triangle PAB is then rotated to the gradient triangle $PA'B'$. Thus, between B' and P, the new line has run m and rise -1, and hence the gradient of the second line is

$$\frac{-1}{m} = -m^{-1}.$$

Key Fact 2.7 Perpendicular Lines

If two lines are perpendicular, then their gradients m_1 and m_2 are such that

$$m_1 m_2 = -1 \qquad \text{or} \qquad m_1 = -\frac{1}{m_2}$$

For Interest

This formula is not true if the lines are parallel to the two coordinate axes, when their gradients are 0 and ∞. There is no difficulty, however, in identifying such perpendicular lines!

It is also true that if two lines have gradients m_1 and m_2 such that $m_1 m_2 = -1$, then these lines are perpendicular. Showing this is a question to be found in Miscellaneous Exercises 1.

Example 2.8.1. *Show that the points $A(0,-5)$, $B(-1,2)$, $C(4,7)$ and $D(5,0)$ form a rhombus.*

We could show this by calculating the four side lengths, but here is an alternative approach. The midpoint of AC is $\left(\frac{1}{2}(0+4), \frac{1}{2}(-5+7)\right) = (2,1)$, while the midpoint of BD is $\left(\frac{1}{2}(-1+5), \frac{1}{2}(2+0)\right) = (2,1)$. Since the midpoints of the two diagonals coincide, the quadrilateral is a parallelogram.

The gradient of AC is $\frac{7-(-5)}{4-0} = 3$, while the gradient of BD is $\frac{0-2}{5-(-1)} = -\frac{1}{3}$. Since $3 \times -\frac{1}{3} = -1$, the two diagonals are perpendicular, and hence $ABCD$ is a rhombus.

Example 2.8.2. *Find the coordinates of the foot of the perpendicular from $A(-2,-4)$ to the line joining $B(0,2)$ and $C(-1,4)$.*

It is often worth drawing a sketch of the problem. It makes it easier to be clear about what needs to be done. The perpendicular from A to the line BC is the line through A that is perpendicular to BC, and its foot is the point where it meets the line BC. The question is asking for the coordinates of P.
First, we calculate that the gradient of BC is $\frac{2-4}{0-(-1)} = -2$. Since its y-intercept is 2, the equation of BC is $y = -2x+2$.
Thus the perpendicular from A has gradient $\frac{1}{2}$ (since $\frac{1}{2} \times -2 = -1$), and so has equation

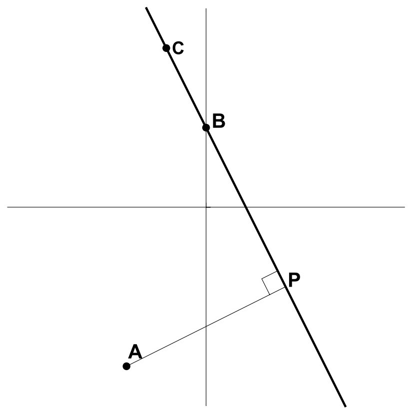


Figure 2.11

$$\begin{aligned} y+4 &= \tfrac{1}{2}(x+2) \\ x-2y &= 6 \end{aligned}$$

Solving the simultaneous equations $2x+y = 2$ and $x-2y = 6$, we deduce that the coordinates of P are $(2,-2)$.

Example 2.8.3. *Find the equation of the perpendicular bisector of the line AB, where A and B have coordinates $(-2,5)$ and $(3,2)$.*

The perpendicular bisector of AB is the line which is perpendicular to AB that passes through the midpoint M of AB.

The midpoint M has coordinates $\left(\frac{1}{2}(-2+3), \frac{1}{2}(5+2)\right) = \left(\frac{1}{2}, \frac{7}{2}\right)$, and the line AB has gradient $\frac{2-5}{3-(-2)} = -\frac{3}{5}$. Thus the perpendicular bisector of AB has gradient $\frac{5}{3}$, and so has equation

$$\begin{aligned} y-\tfrac{7}{2} &= \tfrac{5}{3}\left(x-\tfrac{1}{2}\right) \\ 5x-3y+8 &= 0 \end{aligned}$$

2.9 The Angle Between Two Lines

Using easy trigonometric ideas, the following is clear:

Food For Thought 2.2

The tangent of a straight line is the tangent of the angle that the line makes with the x-axis.

In the diagram on the right, the two lines have gradients $\tan\theta$ and $\tan\phi$ respectively.

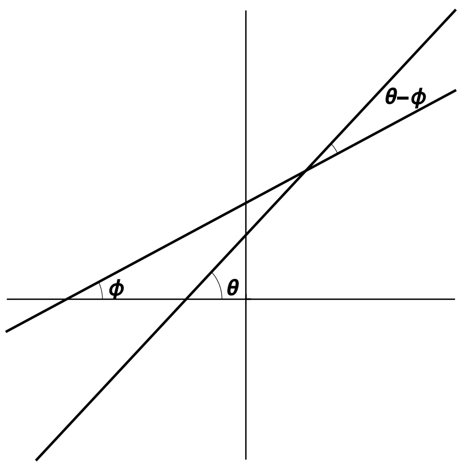


Figure 2.12

The angle between the two lines is $\theta-\phi$, and this can be calculated by elementary means.

Example 2.9.1. *Find the angle between the lines $y = 3x+1$ and $y = x+7$.*

These lines make the angles $\tan^{-1} 3 = 71.6°$ and $\tan^{-1} 1 = 45°$ with the x-axis, and so the angle between them is $71.6° - 45° = 26.6°$.

Although we will need a little more trigonometry to understand why it works, there is a useful formula which can be used to calculate the angle between two lines more directly.

Key Fact 2.8 The Angle Between Two Lines

If two lines have gradients m_1 and m_2, where $m_1 > m_2$, then the angle between these lines is

$$\tan^{-1} \frac{m_1 - m_2}{1 + m_1 m_2}$$

Considering the previous example again, since the lines $y = 3x + 1$ and $y = x + 7$ have gradients 3 and 1 respectively, the angle between them is

$$\tan^{-1} \frac{3-1}{1+3\times 1} = \tan^{-1} \tfrac{1}{2} = 26.6°$$

Note that this formula for the angle between two lines is not properly defined when $m_1 m_2 = -1$, since we are then trying to divide by 0. However, in this case the two lines are perpendicular, and the lack of definition in this formula matches the lack of definition of the tangent of 90°.

2.10 The Distance from a Point to a Line

If P is the foot of the perpendicular from the point A to the line ℓ, and if Q is any other point on the line ℓ, then the distance AQ is the hypotenuse of a right-angled triangle, one of whose other sides is AP. Thus $AQ \geq AP$, and we see that the perpendicular distance AP is the shortest distance from A to any point on the line ℓ. This distance is called, simply, the **distance from the point A to the line** ℓ.
In principle, we know how to calculate this distance. If we know the equation of ℓ and the coordinates of A, then we can find the equation of AP and solve to find the coordinates of P, finally calculating the distance AP. What is interesting is that there is an elegant formula which enables us to avoid all this work!

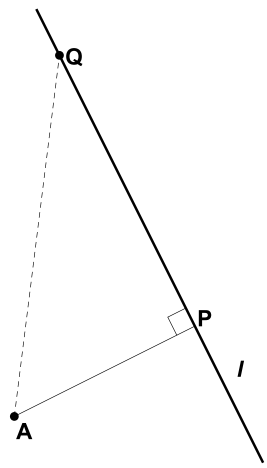


Figure 2.13

Suppose that the line ℓ has equation $ax + by + c = 0$, and that A has coordinates (u, v). Then ℓ has gradient $-\frac{a}{b}$, and so the perpendicular through A has gradient $\frac{b}{a}$, and hence has equation

$$bx - ay = bu - av$$

Solving the equations

$$\begin{aligned} ax + by + c &= 0 \\ bx - ay &= bu - av \end{aligned}$$

simultaneously, we obtain

$$\begin{aligned} (a^2+b^2)x + ac &= a(ax+by+c) + b(bx-ay) = b^2u - abv \\ (a^2+b^2)y + bc &= b(ax+by+c) - a(bx-ay) = a^2v - abu \end{aligned}$$

and so the coordinates of P are

$$\left(\frac{b^2u - abv - ac}{a^2+b^2}, \frac{a^2v - abu - bc}{a^2+b^2} \right)$$

Thus the perpendicular distance d is given by the formula

$$\begin{aligned} d^2 &= \left(u - \frac{b^2u - abv - ac}{a^2+b^2}\right)^2 + \left(v - \frac{a^2v - abu - bc}{a^2+b^2}\right)^2 \\ &= \frac{1}{(a^2+b^2)^2}\left[(a^2u + abv + ac)^2 + (b^2v + abu + bc)^2\right] \\ &= \frac{1}{(a^2+b^2)^2}\left[a^2(au+bv+c)^2 + b^2(au+bv+c)^2\right] \\ &= \frac{(au+bv+c)^2}{a^2+b^2} \end{aligned}$$

To calculate d, we need to take the positive square root of the right-hand side.

Key Fact 2.9 Perpendicular Distance to a Line

The perpendicular distance from a point (u,v) to the line $ax+by+c=0$ is

$$\frac{au+bv+c}{\sqrt{a^2+b^2}}$$

(or minus the above if this expression is negative).

Looking ahead to Chapter 4, we use the modulus function to write this expression as

$$\frac{|au+bv+c|}{\sqrt{a^2+b^2}}$$

Exercise 2C

1. In each part write down the gradient of a line which is perpendicular to one with the given gradient.
   a) 2 *b)* -3 *c)* 4 *d)* $-\frac{5}{6}$
   e) -1 *f)* $1\frac{3}{4}$ *g)* $-\frac{1}{m}$ *h)* m
   i) $\frac{p}{q}$ *j)* 0 *k)* $-m$ *l)* $\frac{a}{b-c}$
2. In each part find the equation of the line through the given point which is perpendicular to the given line. Write your final answer so that it doesn't contain fractions.
   a) $(2,3)$, $y=4x+3$ *b)* $(-3,1)$, $y=-\frac{1}{2}x+3$
   c) $(2,-5)$, $y=-5x-2$ *d)* $(7,-4)$, $y=2\frac{1}{2}$
   e) $(-1,4)$, $2x+3y=8$ *f)* $(4,3)$, $3x-5y=8$
   g) $(5,-3)$, $2x=3$ *h)* $(0,3)$, $y=2x-1$
   i) $(0,0)$, $y=mx+c$ *j)* (a,b), $y=mx+c$
   k) (c,d), $ny-x=p$ *l)* $(-1,-2)$, $ax+by=c$
3. Find the equation of the line through the point $(-2,5)$ which is perpendicular to the line $y=3x+1$. Find also the point of intersection of the two lines.
4. Find the equation of the line through the point $(1,1)$ which is perpendicular to the line $2x-3y=12$. Find also the point of intersection of the two lines.
5. Find the angle between the lines $y=2x-1$ and $y=\frac{1}{2}x+7$ to 2 decimal places.
6. A line through a vertex of a triangle which is perpendicular to the opposite side is called an **altitude**. Find the equation of the altitude through the vertex A of the triangle ABC where A is the point $(2,3)$, B is $(1,-7)$ and C is $(4,-1)$.
7. $P(2,5)$, $Q(12,5)$ and $R(8,-7)$ form a triangle.

a) Find the equations of the altitudes (see Question 6) through R and Q.
b) Find the point of intersection of these altitudes.
c) Show that the altitude through P also passes through this point.

8. The vertices of the triangle PQR are $P(1,5)$, $Q(2,-2)$ and $R(-2,6)$.

a) Write down the equations of the perpendicular bisectors of PQ and QR. Find the point X where these two lines meet.
b) Show that $XP = XQ = XR$.
c) Does X lie on the perpendicular bisector of PR as well?

9*. The vertices of the triangle ABC are $A(a_1,a_2)$, $B(b_1,b_2)$ and $C(c_1,c_2)$, where $a_1^2 + a_2^2 = b_1^2 + b_2^2 = c_1^2 + c_2^2$.

a) Find the equation of the perpendicular bisector of BC. Does this line pass through the origin?
b) Show that the perpendicular bisectors of the three sides of ABC are concurrent.

10*. The vertices of the triangle ABC are $A(a_1,a_2)$, $B(b_1,b_2)$ and $C(c_1,c_2)$, where $a_1^2 + a_2^2 = b_1^2 + b_2^2 = c_1^2 + c_2^2$.

a) Find the equation of the altitude from A.
b) Show that the point $H(a_1+b_1+c_1, a_2+b_2+c_2)$ lies on this altitude.
c) Explain why this shows in general that the three altitudes of a triangle are concurrent. The common point is called the **orthocentre** of the triangle.
d) Show that the origin O, the orthocentre H of the triangle and the centroid G of the triangle (see Question 17 of Exercise 2B) lie in a straight line (are **collinear**).

Chapter 2: Summary

- The length of the line segment AB, where A and B have coordinates (x_1,y_1) and (x_2,y_2) respectively, is

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\,.$$

The gradient of the line segment is

$$\frac{y_2-y_1}{x_2-x_1}$$

and the midpoint of AB has coordinates

$$\left(\tfrac{1}{2}(x_1+x_2), \tfrac{1}{2}(y_1+y_2)\right).$$

- The line with gradient m and y-intercept c has equation

$$y = mx + c\,.$$

The line with gradient m passing through the point (x_1,y_1) has equation

$$y - y_1 = m(x - x_1)\,.$$

The line passing through the two points (x_1,y_1) and (x_2,y_2) has equation

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}\,.$$

- Any line can have its equation written in the form $ax + by + c = 0$ for constants a, b and c. This form is not unique.
- Parallel lines have identical gradients. Lines with gradients m_1 and m_2 are perpendicular when $m_1 m_2 = -1$.
- The distance from the point (u,v) to the straight line with equation $ax + by + c = 0$ is

$$\pm\frac{au+bv+c}{\sqrt{a^2+b^2}}$$

3

Quadratics and Inequalities

In this Chapter we will learn:

- how to complete the square of a quadratic expression, and understand the relationship between the resulting expression and the shape of the graph of the quadratic,
- how to find the discriminant of a quadratic, and understand its relationship to the number of zeros of the quadratic,
- how to solve quadratic equations, and quadratic and linear inequalities,
- how to solve simultaneous equations involving one quadratic equation and one linear one,
- to recognise and solve equations which are quadratic in some function.

3.1 Quadratic Expressions

An equation of the form $bx + c$, where b, c are constants, is called **linear**. Its graph $y = bx + c$ is a straight line. If we add a term in x^2, the resulting expression $ax^2 + bx + c$ is called a **quadratic**, and the corresponding graph $y = ax^2 + bx + c$ is a **parabola**.

Thus a quadratic is an expression of the form $ax^2 + bx + c$, where a, b, c are constants (of course, a must be non-zero: otherwise the expression is linear). Examples of quadratics are $2x^2 - 7x + 3$, $x^2 + 29x$ and $4 - 3x^2$. The constants a, b, c are called **coefficients**: a is the coefficient of x^2, b the coefficient of x, and c is called the constant coefficient.

The graph of a quadratic is always either a 'smile' or a 'frown', with a line of symmetry parallel to the y-axis: the graph is a 'smile' if the coefficient a of x^2 is positive, and a 'frown' otherwise.

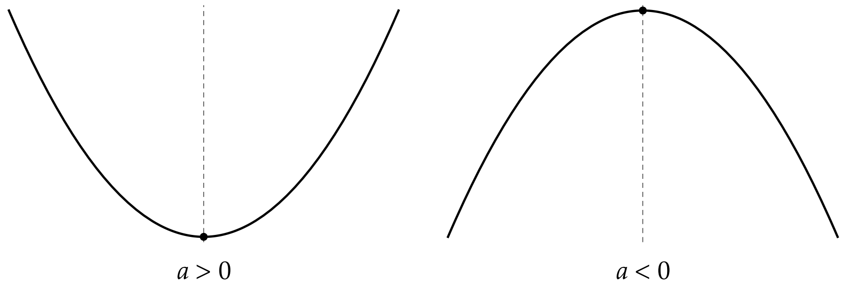


Figure 3.1

The minimum value of a 'smile', and the maximum value of a 'frown' quadratic occurs where the quadratic meets the line of symmetry. Knowing where the line of symmetry is, and knowing the **vertex** of the quadratic (the point where the quadratic meets the line of symmetry) is important, if we are to know the shape of the curve. This can be done by elementary trial-and-error methods.

Example 3.1.1. *Find the line of symmetry and vertex of the quadratic $y = x^2 + 4x - 9$.*

Since $y = -4$ when $x = 1$ and also when $x = -5$, the line of symmetry occurs midway between these two values, so that the line of symmetry has equation $x = -2$.

x	−6	−5	−4	−3	−2	−1	0	1	2
y	3	−4	−9	−12	−13	−12	−9	−4	3

When $x = -2$, $y = -13$, and hence the coordinates of the vertex are $(-2, -13)$.

3.2 Completing The Square

Finding the line of symmetry and the vertex can be done much more effectively by writing the quadratic in an equivalent form. Considering the case of $y = x^2 + 4x - 9$ discussed above, we note that we can write

$$y = x^2 + 4x - 9 = (x+2)^2 - 4 - 9 = (x+2)^2 - 13$$

Since $(x+2)^2$ is a perfect square, it is always at least zero, and it is only zero when $x + 2 = 0$. Thus $y \geq -13$ everywhere, and $y = -13$ precisely when $x = -2$. Thus it is clear that the vertex is $(-2, -13)$, and the line of symmetry is $x = -2$.

Example 3.2.1. *Find the line of symmetry and the vertex of the quadratic* $y = 11 - 2(x+1)^2$.

Since $(x+1)^2$ is a perfect square, $(x+1)^2 \geq 0$, and so $y \leq 11$ for all x, and that $y = 11$ precisely when $x = -1$. Thus the line of symmetry is $x = -1$, and the vertex is $(-1, 11)$.

How do we put a quadratic into this new shape? Start with the simple case where $a = 1$, and consider the quadratic $x^2 + bx + c$. We note that x^2 and bx are the first two terms of the perfect square $\left(x + \frac{1}{2}b\right)^2$. Since $\left(x + \frac{1}{2}b\right)^2 = x^2 + bx + \frac{1}{4}b^2$, we have

$$x^2 + bx = \left(x + \tfrac{1}{2}b\right)^2 - \tfrac{1}{4}b^2$$

and hence

$$x^2 + bx + c = \left(x + \tfrac{1}{2}b\right)^2 + \left(c - \tfrac{1}{4}b^2\right)$$

This process is known as **completing the square**. It can be easily extended to deal with more complex quadratics.

Example 3.2.2. *Complete the square for these quadratics:*
(a) $x^2 + 10x + 32$, *(b)* $2x^2 + 10x + 7$, *(c)* $3 - 4x - 2x^2$

(a) $x^2 + 10x + 32 = (x+5)^2 - 25 + 32 = (x+5)^2 + 7$

(b) Start by taking out the factor of 2 in front of x^2, so that

$$\begin{aligned} 2x^2 + 10x + 7 &= 2(x^2 + 5x) + 7 = 2\left[\left(x + \tfrac{5}{2}\right)^2 - \tfrac{25}{4}\right] + 7 \\ &= 2\left(x + \tfrac{5}{2}\right)^2 - \tfrac{25}{2} + 7 = 2\left(x + \tfrac{5}{2}\right)^2 - \tfrac{11}{2} \end{aligned}$$

(c) Do the same with the factor of -2, so

$$\begin{aligned} 3 - 4x - 2x^2 &= 3 - 2(x^2 + 2x) = 3 - 2\left[(x+1)^2 - 1\right] \\ &= 3 - 2(x+1)^2 + 2 = 5 - (x+1)^2 \end{aligned}$$

where we have completed the square for $x^2 + 2x$ to obtain $(x+1)^2 - 1$.

Example 3.2.3. *Complete the square for* $x^2 - 3x + 1$*; use the result to solve the equation* $x^2 - 3x + 1 = 0$.

Since $x^2 - 3x + 1 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 1 = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$, we solve the equation

$$\begin{aligned} x^2 - 3x + 1 &= 0 \\ \left(x - \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + 1 &= 0 \\ \left(x - \tfrac{3}{2}\right)^2 &= \tfrac{5}{4} \\ x - \tfrac{3}{2} &= \pm\tfrac{1}{2}\sqrt{5} \\ x &= \tfrac{1}{2}(3 \pm \sqrt{5}) \end{aligned}$$

Example 3.2.4. *Complete the square for $12x^2 - 7x - 12$, and use the result to factorise the quadratic.*

We have

$$\begin{aligned} 12x^2 - 7x - 12 &= 12\left[x^2 - \tfrac{7}{12}x\right] - 12 \;=\; 12\left[\left(x - \tfrac{7}{24}\right)^2 - \tfrac{49}{576}\right] - 12 \\ &= 12\left(x - \tfrac{7}{24}\right)^2 - \tfrac{49}{48} - 12 \;=\; 12\left(x - \tfrac{7}{24}\right)^2 - \tfrac{625}{48} \end{aligned}$$

and so, using the 'Difference of Two Squares' technique,

$$\begin{aligned} 12x^2 - 7x - 12 &= 12\left[\left(x - \tfrac{7}{24}\right)^2 - \tfrac{625}{576}\right] \;=\; 12\left[\left(x - \tfrac{7}{24}\right)^2 - \left(\tfrac{25}{24}\right)^2\right] \\ &= 12\left(x - \tfrac{7}{24} - \tfrac{25}{24}\right)\left(x - \tfrac{7}{24} + \tfrac{25}{24}\right) \;=\; 12\left(x - \tfrac{4}{3}\right)\left(x + \tfrac{3}{4}\right) \\ &= (3x - 4)(4x + 3) \end{aligned}$$

To sum up:

Key Fact 3.1 Completing the Square

$$\begin{aligned} x^2 + bx + c &= \left(x + \tfrac{1}{2}b\right)^2 + c - \tfrac{1}{4}b^2 \\ ax^2 + bx + c &= a\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2 - 4ac}{4a} \end{aligned}$$

It is probably just as easy to understand the process for completing the square in general, and to work out each case ' 'by hand', as to learn these formulae!

Exercise 3A

1. Find the vertex and the line of symmetry for each of the following:
   a) $y = (x-2)^2 + 3$ *b)* $y = (x-5)^2 - 4$ *c)* $y = (x+3)^2 - 7$
   d) $y = (2x-3)^2 + 1$ *e)* $y = (5x+3)^2 + 2$ *f)* $y = (3x+7)^2 - 4$
   g) $y = (x-3)^2 + c$ *h)* $y = (x-p)^2 + q$ *i)* $y = (ax+b)^2 + c$

2. Find the least (or, if appropriate, the greatest) value of each of the following quadratic expressions, and the value of x for which this value occurs.
   a) $(x+2)^2 - 1$ *b)* $(x-1)^2 + 2$ *c)* $5 - (x+3)^2$
   d) $(2x+1)^2 - 7$ *e)* $3 - 2(x-4)^2$ *f)* $(x+p)^2 + q$
   g) $(x-p)^2 - q$ *h)* $r - (x-t)^2$ *i)* $c - (ax+b)^2$

3. Solve the following quadratic equations, leaving surds in your answer.
   a) $(x-3)^2 - 3 = 0$ *b)* $(x+2)^2 - 4 = 0$ *c)* $2(x+3)^2 = 5$
   d) $(3x-7)^2 = 8$ *e)* $(x+p)^2 - q = 0$ *f)* $a(x+b)^2 - c = 0$

4. Complete the square for the following quadratic expressions:
   a) $x^2 + 2x + 2$ *b)* $x^2 - 8x - 3$ *c)* $x^2 + 3x - 7$
   d) $5 - 6x + x^2$ *e)* $x^2 + 14x + 49$ *f)* $2x^2 + 12x - 5$
   g) $3x^2 - 12x + 3$ *h)* $7 - 8x - 4x^2$ *i)* $2x^2 + 5x - 3$

5. By completing the square, factorise the following expressions:
   a) $x^2 - 2x - 35$ *b)* $x^2 - 14x - 176$ *c)* $x^2 + 6x - 432$
   d) $6x^2 - 5x - 6$ *e)* $14 + 45x - 14x^2$ *f)* $12x^2 + x - 6$

6. By completing the square, find (as appropriate) the least or greatest value of each of the following expressions, and the value of x for which this occurs.
   a) $x^2 - 4x + 7$ *b)* $x^2 - 3x + 5$ *c)* $4 + 6x - x^2$
   d) $2x^2 - 5x + 2$ *e)* $3x^2 + 2x - 4$ *f)* $3 - 7x - 3x^2$

3.3 Solving Quadratic Equations

We now have three methods that can be used to solve quadratic equations:

- Factorisation is often the simplest method. To solve $x^2-6x+8 = 0$ we write $(x-2)(x-4) = 0$. Thus either $x-2=0$ or $x-4=0$, so that $x=2$ or $x=4$. The numbers 2 and 4 are the **roots** of the equation. This method might be impossible, or just difficult, to do: try finding the factors of $30x^2-11x-30$, for example.
- We can use the so-called **Quadratic Formula**: the roots of the equation $ax^2+bx+c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

- Quadratics can also be solved by completing the square. The equation $x^2+2x-4 = 0$ becomes $(x+1)^2-5=0$, and hence $(x-1)^2=5$, so that $x-1=\pm\sqrt{5}$, and hence $x = 1\pm\sqrt{5}$.

It is worth noting that the Quadratic Formula is actually derived by completing the square! We see that

$$\begin{aligned} ax^2+bx+c &= 0 \\ a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a} &= 0 \\ \left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2} \\ x+\frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{2a} \\ x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \end{aligned}$$

Of course, for the Quadratic Formula (or the general method of completing the square) to work we need $a \neq 0$. Since the quadratic equation becomes a linear one (and thus easy to solve) when $a=0$, this is not a demanding restriction!

If a quadratic equation cannot be solved by simple factorisation, it is likely that the roots will involve surds. Always try to express solutions **exactly**, using surds. It is far better to say that the roots of an equation are $\frac{1}{2}(1\pm\sqrt{2})$, instead of saying that the roots are 1.21 and −0.21 to 2 decimal places. Any result giving an answer to a fixed number of decimal places will be an approximation, and using that approximation in later calculations will probably produce errors. It is better to be exact whenever possible.

For Interest

There is a school of thought that says that if you can do a problem on a calculator, you are not doing mathematics, but just sums. While there are problems where calculators are necessary, try to adopt the strategy of not using your calculator as a first resort!

It is important to realise that there are some quadratics which we cannot (as yet) solve! It is quite possible to choose constants a, b and c such that b^2-4ac is negative, in which case the Quadratic Formula cannot be implemented (or at least not until we have learned about Complex Numbers).

Example 3.3.1. *Solve the equations:*
(*a*) $2x^2-3x-4 = 0$ (*b*) $2x^2-3x+4 = 0$ (*c*) $30x^2-11x-30 = 0$

(a) Using the formula

$$x = \frac{3\pm\sqrt{(-3)^2-4\times2\times(-4)}}{4} = \frac{1}{4}\left[3\pm\sqrt{41}\right]$$

(b) Completing the square

$$\begin{aligned} 2\left(x-\frac{3}{4}\right)^2 - \frac{9}{8}+4 &= 0 \\ \left(x-\frac{3}{4}\right)^2 &= -\frac{23}{16} \end{aligned}$$

Since the right-hand side is negative, we cannot take its square root, and hence there are no solutions. Had we used the formula, we would have been asked to calculate $\sqrt{-23}$, which would indicate that there was no solution.

(c) Completing the square

$$\begin{aligned}30\left(x^2 - \tfrac{11}{30}x\right) &= 30\\ x^2 - \tfrac{11}{30}x &= 1\\ \left(x - \tfrac{11}{60}\right)^2 &= 1 + \tfrac{121}{3600} = \tfrac{3721}{3600}\\ x - \tfrac{11}{60} &= \pm\tfrac{61}{60}\\ x &= \tfrac{72}{60} = \tfrac{6}{5}, -\tfrac{50}{60} = -\tfrac{5}{6}\end{aligned}$$

Having found the solutions we observe that the equation **could** have been solved by finding the elusive factorisation $(5x-6)(6x+5) = 0$.

3.4 The Discriminant

Considering the quadratic formula, the expression b^2-4ac inside the square root has a particular importance.

- If $b^2 - 4ac > 0$, then there is no problem evaluating both roots of the quadratic, and the quadratic equation $ax^2 + bx + c = 0$ has two distinct roots.
- If $b^2 - 4ac = 0$, then the formula tells us that $x = -\frac{b}{2a}$ is the single root of the equation. In a sense that will be made clear later, this root can be regarded as occurring twice, and is often called a **repeated root**.
- If $b^2 - 4ac < 0$, then the quadratic formula fails to give any solutions, and hence there are no solutions.

Key Fact 3.2 The Discriminant

The quantity $\Delta = b^2 - 4ac$ is called the **discriminant** of the quadratic $ax^2 + bx + c$, since it discriminates between the possible types of solution of the equation $ax^2 + bx + c = 0$: the equation $ax^2 + bx + c = 0$ has two distinct roots if $\Delta > 0$, has one (repeated) root if $\Delta = 0$, and has no real roots if $\Delta < 0$.
Moreover, if a, b, c are integers and $\Delta > 0$ is a perfect square, then the roots of the quadratic equation will be rational. In this case, a simple factorisation of the quadratic is possible.

Example 3.4.1. *What can be said about the roots of the following quadratic equations?*

a) $2x^2 - 3x - 4 = 0$ *b)* $2x^2 - 3x - 5 = 0$
c) $2x^2 - 4x + 5 = 0$ *d)* $2x^2 - 4x + 2 = 0$.

(a) Since $\Delta = (-3)^2 - 4\times 2\times(-4) = 41 > 0$ is not a perfect square, the equation has two irrational roots.

(b) This time $\Delta = (-3)^2 - 4\times 2\times(-5) = 49 = 7^2 > 0$, and hence the equation has two rational roots.

(c) This time $\Delta = (-4)^2 - 4\times 2\times 5 = -24 < 0$, and hence the equation has no roots.

(d) Finally $\Delta = (-4)^2 - 4\times 2\times 2 = 0$, so the equation has one (repeated) root.

Example 3.4.2. *The equation $kx^2 - 2x - 7 = 0$ has two real roots. What can be said about the value of k?*

The quadratic has discriminant $4 + 28k$. Since the equation has two real roots, $\Delta > 0$, and hence $k > -\frac{1}{7}$.

Example 3.4.3. *The equation $3x^2 + 2x + k = 0$ has a repeated root. Find the value of k.*

The quadratic has discriminant $4-12k$. Since this must be equal to 0, it follows that $k=\frac{1}{3}$.

Note that in none of these cases was it necessary to find the roots: determining the discriminant was enough.

Exercise 3B

1. Solve the following equations, where possible, giving exact answers.

   a) $x^2+3x-5=0$ *b)* $x^2-4x-7=0$ *c)* $x^2+6x+9=0$
   d) $2x^2+7x+3=0$ *e)* $8-3x-x^2=0$ *f)* $x^2+x+1=0$

2. Use the discriminant to determine whether the following equations have two roots, one root or no roots. The constants p and q are positive.

   a) $x^2-3x-5=0$ *b)* $x^2+2x+1=0$ *c)* $x^2-3x+4=0$
   d) $3x^2-6x+5=0$ *e)* $x^2+px-q=0$ *f)* $x^2-px+p^2=0$

3. The following equations have the number of roots shown in brackets. Deduce as much as you can about the value of k.

   a) $x^2+3x+k=0\ (2)$ *b)* $x^2-7x+k=0\ (1)$ *c)* $kx^2-3x+5=0\ (0)$
   d) $3x^2+5x-k=0\ (2)$ *e)* $x^2-4x+3k=0\ (1)$ *f)* $kx^2-5x+7=0\ (0)$
   g) $x^2-kx+4=0\ (2)$ *h)* $x^2+kx+9=0\ (0)$ *i)* $kx^2+7x+k=0\ (1)$.

4. If a and c are both positive, what can be said about the graph of $y=ax^2+bx-c$?

5. If a is negative and c is positive, what can be said about the graph of $y=ax^2+bx+c$?

6. If α and β are the roots of the quadratic equation $x^2+bx+c=0$, express $\alpha+\beta$ and $\alpha\beta$ in terms of b and c (Hint: Factorise the quadratic).

7. If α and β are the roots of the quadratic equation $ax^2+bx+c=0$, express $\alpha+\beta$ and $\alpha\beta$ in terms of a, b and c.

8. Show that the equation $\frac{x+1}{x-4}=\frac{x+7}{2x+5}$ has no real roots.

9. Show that there is no real value of p for which the equation

$$(p+1)x^2+(2p+1)x+9=0$$

   has a repeated root. How many solutions does this equation have, for any value of p?

10*. Find the only values of p and q such that the equation $x^2+px=q^2$ has a unique solution.

11*. Given that the roots of the equation $x^2+ux+(u+5)=0$ differ by 1, find the possible values of u, and the roots of the equation for each value of u.

3.5 Simultaneous Equations

Suppose we want to solve the simultaneous equations

$$y = x^2 \qquad x+y = 6\,.$$

The standard technique is to use the linear expression to express either x or y in terms of the other variable, and substitute this expression into the quadratic equation.

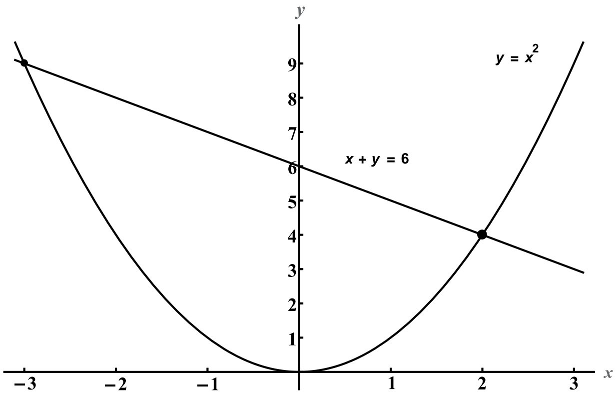


Figure 3.2

In this case we can write $y = 6 - x$, and substitution gives us

$$\begin{aligned} 6 - x &= x^2 \\ x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \end{aligned}$$

so that $x = -3, 2$. Substituting these values for x into the linear expression gives the corresponding values of y. They are $y = 9, 4$. Thus there are two solutions: either $x = 2, y = 4$ or $x = -3, y = 9$. Note that the x and y values go together in pairs: $x = 2$ corresponds to $y = 4$ and not to $y = 9$. This correspondence can be seen in the graph, where the coordinates of the two points of intersection of the parabola $y = x^2$ and the line $x + y = 6$ are $(2, 4)$ and $(-3, 9)$.

Example 3.5.1. *Solve the simultaneous equations* $x^2 - 2xy + 3y^2 = 11$ *and* $x - 3y = 1$.

The linear expression can be written $x = 3y + 1$. We substitute:

$$\begin{aligned} x^2 - 2xy + 3y^2 &= 11 \\ (3y+1)^2 - 2y(3y+1) + 3y^2 &= 11 \\ 9y^2 + 6y + 1 - 6y^2 - 2y + 3y^2 &= 11 \\ 6y^2 + 4y - 10 &= 0 \\ 3y^2 + 2y - 5 &= 0 \\ (3y+5)(y-1) &= 0 \end{aligned}$$

so that $y = 1, -\frac{5}{3}$. The corresponding values of x are $4, -4$. Thus the solutions are $x = -4, y = -\frac{5}{3}$ and $x = 4, y = 1$.

Example 3.5.2. *At how many points does the line* $x + 2y = 3$ *meet the curve* $2x^2 + y^2 = 4$*?*

Substituting $x = 3 - 2y$ into the other equation, we obtain

$$\begin{aligned} 2(3-2y)^2 + y^2 &= 4 \\ 9y^2 - 24y + 14 &= 0 \end{aligned}$$

Since this quadratic has discriminant $(-24)^2 - 4 \times 9 \times 14 = 72$, this quadratic has two solutions. Thus the line meets the curve at two points.

3.6 Disguised Quadratics

Sometimes you will meet equations which are not, at first sight, quadratics. However, these equations can be turned into quadratics by careful algebra and a sensible substitution.

Example 3.6.1. *Solve the equations*
(a) $x = 2 + 8x^{-1}$, *(b)* $t^4 - 13t^2 + 36 = 0$, *(c)* $\sqrt{x} = 6 - x$.

(a) Multiplying the equation by x yields $x^2 = 2x + 8$, or $x^2 - 2x - 8 = 0$. This factorises to read $(x-4)(x+2) = 0$, and hence $x = 4, -2$.

(b) Substituting $x = t^2$ yields $x^2 - 13x + 36 = 0$, or $(x-4)(x-9) = 0$, so that $x = 4, 9$. Thus $t^2 = 4, 9$ so that $t = \pm 2, \pm 3$.

(c) Substituting $\sqrt{x} = y$ yields $y = 6 - y^2$, so that $y^2 + y - 6 = 0$, and hence $(y+3)(y-2) = 0$, so that $y = -3, 2$. But since $y = \sqrt{x}$ **must** be positive we must have $y = 2$, so that $\sqrt{x} = 2$, and hence $x = 4$.

An alternative approach would be to square both sides, yielding $x = (6-x)^2 = 36 - 12x + x^2$, and hence $x^2 - 13x + 36 = 0$. Solving this as above, we deduce that $x = 4, 9$. Where has this (incorrect) 'solution' of $x = 9$ come from? The problem is that the squared equation $x = (6-x)^2$ is satisfied by solutions to the equation $-\sqrt{x} = 6 - x$, as well as by solutions to the original equation. Squaring frequently results in an equation which is satisfied by more than the solutions of the initial equation. The only case when this is not true is when both sides of the equation are known to be positive (or both negative); keeping this in mind is an added complication which can be avoided by adopting the first approach.

Exercise 3C

1. Solve the following pairs of simultaneous equations:

a) $y = x + 1, \quad x^2 + y^2 = 25$
b) $x + y = 7, \quad x^2 + y^2 = 25$
c) $2x + y = 5, \; x^2 + y^2 = 25$
d) $y = 1 - x, \quad y^2 - xy = 0$
e) $y = 2x + 1, \; y = x^2 - x + 3$
f) $y = 3x + 2, \; x^2 + y^2 = 26$
g) $y = 2x - 12, \; x^2 + 4xy - 3y^2 = -27$
h) $2x - 5y = 6, \; 2xy - 4x^2 - 3y = 1$

2. Find the number of points of intersection of the straight line with the curve in each case:

a) $y = 1 - 2x, \; x^2 + y^2 = 1$
b) $y = \frac{1}{2}x - 1, \; y = 4x^2$
c) $y = 3x - 1, \; xy = 12$
d) $4y - x = 16, \; y^2 = 4x$
e) $3y - x = 15, \; 4x^2 + 9y^2 = 36$
f) $4y = 12 - x, \; xy = 9$

3*. Solve the simultaneous equations

$$x^2 + y^2 + 2x - 4y = 0 \qquad\qquad x^2 + y^2 + 5x - 3y = 6$$

4. Solve the following equations exactly:

a) $x^4 - 5x^2 + 4 = 0$
b) $x^4 - 10x^2 + 9 = 0$
c) $x^6 - 7x^3 - 8 = 0$
d) $x^6 + x^3 = 12$
e) $x = 3 + 10x^{-1}$
f) $2t + 5 = \frac{3}{t}$
g) $x = \frac{12}{x+1}$
h) $\sqrt{t}(\sqrt{t} - 6) = -9$
i) $x - \frac{2}{x+2} = \frac{1}{3}$
j) $\frac{12}{x+1} - \frac{10}{x-3} = -3$
k) $\frac{15}{2x+1} + \frac{10}{x} = \frac{55}{2}$
l) $\frac{1}{y^2} - \frac{1}{y^2+1} = \frac{1}{2}$
m) $x - 8 = 2\sqrt{x}$
n) $x + 15 = 8\sqrt{x}$
o) $t - 5\sqrt{t} - 14 = 0$
p) $\sqrt[3]{x^2} - \sqrt[3]{x} - 6 = 0$
q) $\sqrt[3]{t^2} - 3\sqrt[3]{t} = 4$

5. Solve the equation $(x^2 + 2)^2 - 14(x^2 + 2) + 33 = 0$

6. Solve the equation $x(x+2) + \frac{24}{x(x+2)} = 11$.

7*. By substituting $y = x - x^{-1}$ show that the expression

$$6x^4 - 25x^3 + 12x^2 + 25x + 6$$

can be written as $x^2(6y^2 - 25y + 24)$. Hence solve the quartic equation

$$6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$$

8*. Solve the equation $(7 + 4\sqrt{3})^x + (7 - 4\sqrt{3})^x = 4$.

3.7 Sketching Quadratics

We frequently want to know the shape of a curve. Drawing a graph inexactly, while showing enough information to indicate its key properties, is called **sketching** the graph. When sketching straight line graphs, it is generally enough to indicate the x- and y-intercepts (the points where the line meets the x- and y-axes, or perhaps (qualitatively) the gradient of the curve:

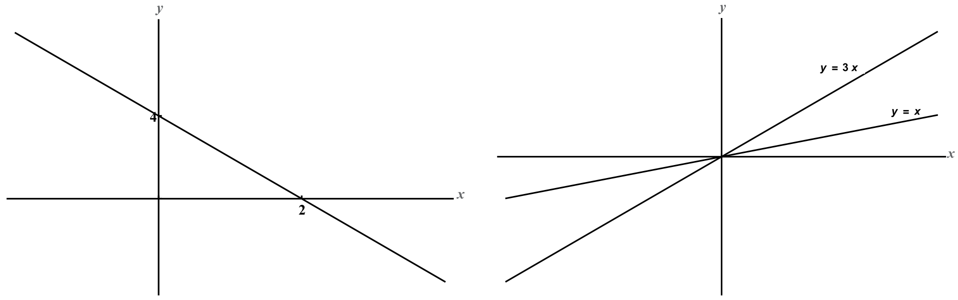
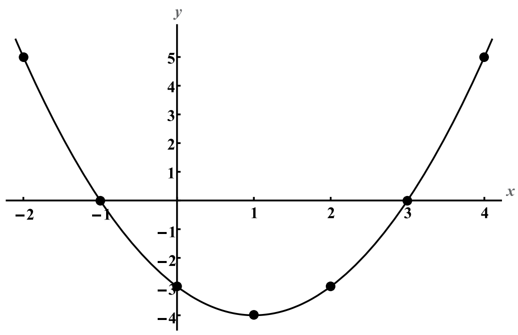


Figure 3.3

We certainly do not need to plot individual points on the curve, as if we were drawing the graph on graph paper! The same thing is true when we want to sketch a quadratic curve: we are not being asked to draw an accurate graph, and do not need to plot a large number of individual points:



This is a plot, not a sketch!

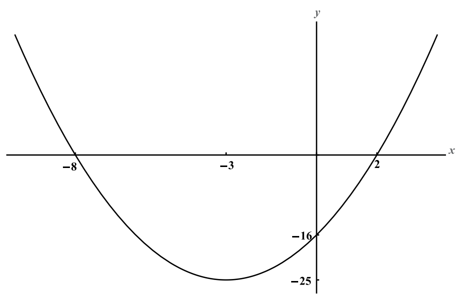
Figure 3.4

A sketch of a quadratic should include the key essentials of the curve, which can be determined by factorisation or completing the square:

- Is it a 'smile' or a 'frown'?
- Where does it intercept the axes?
- Where is the vertex?

Example 3.7.1. *Sketch the quadratic* $y = x^2 + 6x - 16$.

Since $y = (x+8)(x-2) = (x+3)^2 - 25$, the quadratic is a 'smile' with x-intercepts 2 and -8 and a y-intercept -16, with vertex $(-3, -25)$.



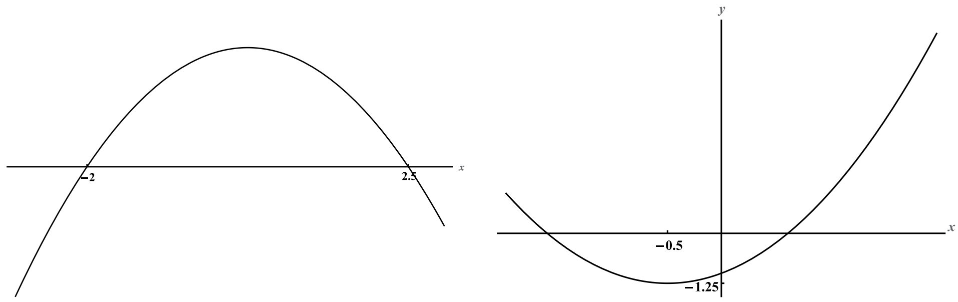
$y = x^2 + 6x - 1$

Figure 3.5

Sometimes it is not even necessary to provide this much information. We might only be interested in the sketch for the information about when the function is positive or negative, or we might only be interested in when the function has a positive gradient and when it has a negative one. You need to decide from context what information is needed in your sketch.

Example 3.7.2. *Sketch the curve $y = 10 + x - 2x^2$, and hence determine the range of values of x for which the curve is above the x-axis.*

We want to know the sign of y, so shall not include information about the vertex. Since $y = (5 - 2x)(2 + x)$, there are x-intercepts at -2 and $\frac{5}{2}$. An adequate sketch can omit the y-axis. The graph lies above the x-axis for $-2 < x < \frac{5}{2}$.



$y = 10 + x - x^2$ $\qquad$ $y = x^2 + x - 1$

Figure 3.6

Example 3.7.3. *Sketch the curve $y = x^2 + x - 1$.*

Since $y = \left(x + \frac{1}{2}\right)^2 - \frac{5}{4}$, the vertex occurs at $\left(-\frac{1}{2}, -\frac{5}{4}\right)$. The intercepts at $\frac{1}{2}[-1 \pm \sqrt{5}]$ are a little too complicated to add to a small sketch (the question does not make it plain that they are needed!)

These ideas can be used in reverse to retrieve the quadratic from its key properties.

Example 3.7.4. *Write down the equation of the quadratic with vertex $(1, 2)$ which passes through $(3, -6)$.*

The quadratic must have equation $y = a(x - 1)^2 + 2$ for some a, since $-6 = a(3-1)^2 + 2 = 2 + 4a$, we deduce that $a = -2$, so that $y = -2(x-1)^2 + 2 = 4x - 2x^2$.

Exercise 3D

1. Sketch the following graphs, showing all intercepts with the axes and the vertex (the constants a and b are positive, with $b > a$):
   a) $y = x^2 - 6x + 8$ *b)* $y = 3 - 2x - x^2$ *c)* $y = x^2 - 2x$
   d) $y = 2x^2 - 5x - 3$ *e)* $y = -3(x - 4)^2$ *f)* $y = 12 + x - x^2$
   g) $y = x^2 + 2ax$ *h)* $y = (x - a)^2 - b^2$
2. By sketching the graphs, determine the range of values of x for which each graph is below the x-axis:
   a) $y = x^2 + 2x - 1$ *b)* $y = 7 - 6x - x^2$ *c)* $y = x^2 + 8x + 16$
3. Find the equation of the parabola which:
   a) crosses the x-axis at $(1, 0)$ and $(5, 0)$, and crosses the y-axis at $(0, 15)$,
   b) crosses the x-axis at $(-2, 0)$ and $(7, 0)$ and crosses the y-axis at $(0, -56)$,
   c) passes through the points $(-6, 0)$, $(-2, 0)$ and $(0, -6)$,
   d) has a minimum value at $(1, 3)$ and passes through $(4, 39)$,
   e) passes through the points $(1, 3)$, $(5, 3)$ and $(2, 6)$

4*. What is the equation of the quadratic passing through the points $(1, 0)$, $(2, 5)$ and $(6, 45)$?

5*. Use your sketch of $y = x^2 - 6x + 8$ from Question 1 to sketch $y = (x + k)^2 - 6(x + k) + 8$ for $2 < k < 4$.

3.8 Solving Inequalities

3.8.1. Notation and Algebra

We often want to compare one number with another and say which is the greater. This comparison is expressed by using the inequality symbols $>$, $<$, $\leq$ and $\geq$. These symbols should already be familiar.

- The symbol $a < b$ means that a is less than b,
- the symbol $a > b$ means that a is greater than b,
- the symbol $a \leq b$ means that a is less than or equal to b,
- the symbol $a \geq b$ means that a is greater than or equal to b.

Thus $a < b$ and $b > a$ mean the same thing. Similarly, $a \leq b$ and $b \geq a$ mean the same thing.

The symbols $<$ and $>$ are called **strict** inequalities, while the symbols $\leq$ and $\geq$ are called **non-strict** inequalities.

We often wish to manipulate algebraic inequalities. When solving algebraic equations, we use a number of techniques, which boil down to 'do the same thing to both sides of the equation'. We have almost the same amount of freedom with inequalities.

The rules for handling inequalities are easy to write down, but since they differ subtly from those for equations, it is worth taking the time to see why they work. The simplest way to do this is to note that all inequality statements are equivalent to statements about positive numbers:

$$\begin{array}{rcl} a < b & \Leftrightarrow & b - a > 0 \\ a \leq b & \Leftrightarrow & b - a \geq 0 \end{array}$$

and the positive numbers have the property that they remain positive when added together or multiplied together:

$$\begin{array}{rcl} c, d > 0 & \Rightarrow & c + d, cd > 0 \\ c, d \geq 0 & \Rightarrow & c + d, cd \geq 0 \end{array}$$

From these observations we can deduce that following:

Key Fact 3.3 Operations on Inequalities

$$\begin{array}{rclr} a < b & \Rightarrow & a + c < b + c & \text{for any real } c \\ a \leq b & \Rightarrow & a + c \leq b + c & \text{for any real } c \\ a < b & \Rightarrow & ad < bd & \text{for any } d > 0 \\ a \leq b & \Rightarrow & ad \leq bd & \text{for any } d > 0 \\ a < b & \Rightarrow & ae > be & \text{for any } e < 0 \\ a \leq b & \Rightarrow & ae \geq be & \text{for any } e > 0 \end{array}$$

If $a < b$, then $(b+c)-(a+c) = b-a > 0$ and hence $a+c < b+c$. The second inequality follows similarly. If $a < b$, then $b - a > 0$, and hence (for $d > 0$) $bd - ad = (b-a)d > 0$, so that $ad < bd$. The fourth inequality follows similarly. The last two need care, since they state that when an inequality is multiplied by a **negative** number, the sense of the inequality needs to be reversed. To see why this is so, we note that $-e > 0$ when $e < 0$. Thus

$$a < b \quad \Rightarrow \quad -ae = (-e)a < (-e)b = -be \quad \Rightarrow \quad ae - be > 0 \quad \Rightarrow \quad ae > be$$

Thus we can manipulate inequalities in almost exactly the same way that we manipulate equations: we can add or subtract numbers to both sides, and can multiply both sides by positive numbers. We just have to remember that multiplying both sides of an inequality by a negative number reverses the sense of that inequality.

3.8.2. Solving Linear Inequalities

Solving many inequalities is simply a matter of exploiting the above rules, and using standard algebra.

Example 3.8.1. *Solve the inequality* $\frac{1}{3}(4x+3) - 3(2x-4) \geq 20$.

We successively multiply both sides of the inequality by 3, subtract 39 from both sides, and divide both sides by -14 to obtain the answer.

$$\begin{aligned} \tfrac{1}{3}(4x+3)-3(2x-4) &\geq 20 \\ 4x+3-9(2x-4) &\geq 60 \\ 4x+3-18x+36 &\geq 60 \\ -14x+39 &\geq 60 \\ -14x &\geq 60-39=21 \\ x &\leq \tfrac{21}{-14}=-\tfrac{3}{2} \end{aligned}$$

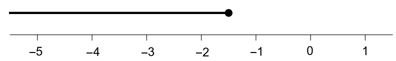


Figure 3.7

The series of operations performed is the same as would have been performed had we been asked to solve the equation $\frac{1}{3}(3x-4)-3(2x-4) = 20$. The only time we need to take care is when multiplying by a number; the sign of that number has to be considered, since it affects the sense of the inequality.

Exercise 3E

Solve the following inequalities.

1. *a)* $x-3>11$ *b)* $x+7<11$ *c)* $2x+3\leq 8$ *d)* $3x-5\geq 16$
   e) $3x+7>-5$ *f)* $5x+6\leq -10$ *g)* $2x+3<-4$ *h)* $3x-1\leq -13$

2. *a)* $\frac{x+3}{2}>5$ *b)* $\frac{x-4}{6}\leq 3$ *c)* $\frac{2x+3}{4}<-5$ *d)* $\frac{3x+2}{5}\leq 4$
   e) $\frac{4x-3}{2}\geq -7$ *f)* $\frac{5x+1}{3}>-3$ *g)* $\frac{3x-2}{8}<1$ *h)* $\frac{4x-2}{3}\geq -6$

3. *a)* $x-4\leq 5+2x$ *b)* $x-3\geq 5-x$ *c)* $2x+5<4x-7$
   d) $3x-4>5-x$ *e)* $4x\leq 3(2-x)$ *f)* $3x\geq 5-2(3-x)$
   g) $6x<8-2(7+x)$ *h)* $5x-3>x-3(2-x)$ *i)* $6-2(x+1)\leq 3(1-2x)$

4. *a)* $\frac{1}{3}(8x+1)-2(x-3)>10$ *b)* $\frac{5}{2}(x+1)-2(x-3)<7$
   c) $\frac{1}{4}(x+1)+\frac{1}{6}\geq\frac{1}{3}(2x-5)$ *d)* $\frac{1}{2}x-\frac{1}{5}(3-2x)\leq 1$

3.8.3. Quadratic Inequalities

Solving inequalities involving quadratics requires us to determine where some quadratic expression is positive or negative. The easiest way of handling these inequalities involves factorisation.

Example 3.8.2. *Solve the inequality* $x^2-6x+8<0$.

We sketch the curve of $y=x^2-6x+8$. Since $y=(x-2)(x-4)$, we see that the graph crosses the x-axis at $x=2$ and $x=4$. Since the curve is a 'smile', the vertex of the curve lies below the x-axis, and we obtain the sketch.
We now need to find on the graph where $y<0$. This occurs when x lies between 2 and 4, so when both $x>2$ and $x<4$.

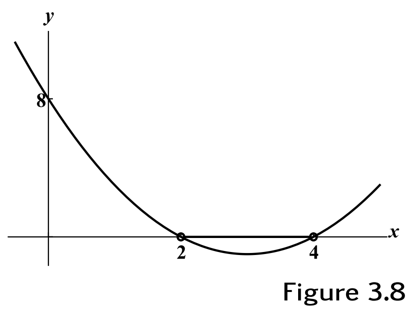


Figure 3.8

Note that, since $2<4$, it is acceptable to put these two inequalities together and write the solution as $2<x<4$. A solution set of this type is called an **interval**.

Example 3.8.3. *Solve the inequality* $x^2\leq 2x+3$.

We sketch the curve of $y = x^2 - 2x - 3$. Since $y = (x+1)(x-3)$, we see that the graph crosses the x-axis at $x = -1$ and $x = 3$. Since the curve is a 'smile', the vertex of the curve lies below the x-axis, and we obtain the sketch.
We now need to find on the graph where $y \le 0$. This occurs when either $x \le -1$ or $x \ge 3$.

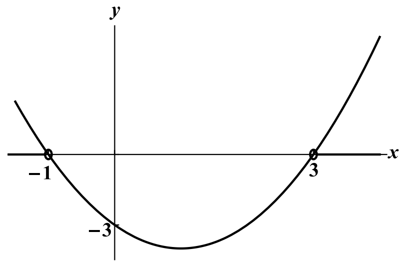


Figure 3.9

However, since it is not true that $3 \le -1$, we cannot put these two inequalities together and write $3 \le x \le -1$. Another reason why we cannot do this is that the two inequalities cannot be true at the same time — either one is true or the other. Writing a single inequality implies that both inequalities are intended to be true at the same time, as was the case in the previous example.

An alternative approach can solve these problems without sketching a graph, by simply determining the sign of the function for key values of x. This method is not as important in this case, since we know the shapes of quadratic graphs, but will be very useful when we want to consider more complex inequalities, involving cubic or worse expressions.

Example 3.8.4. *Solve the inequality* $5 + 4x - x^2 \le 0$.

Since $5 + 4x - x^2 = (5-x)(1+x)$, the function equals zero at $x = 5$ and $x = -1$. These values of 5 and −1 are called the **critical values** of the function. Make a table showing the signs of the factors in the product $(5-x)(1+x)$.

	$\mathbf{x < -1}$	$\mathbf{x = -1}$	$\mathbf{-1 < x < 5}$	$\mathbf{x = 5}$	$\mathbf{x > 5}$
$\mathbf{5 - x}$	+	+	+	0	−
$\mathbf{1 + x}$	−	0	+	+	+
$\mathbf{5 + 4x - x^2}$	−	0	+	0	−

We see that the solution is $x \le -1$ or $x \ge 5$ (the critical values are included, due to the non-strict inequality).

Example 3.8.5. *If* $a > 0$, *solve the inequality* $x^2 \le a^2$.

We need to solve $(x-a)(x+a) = x^2 - a^2 \le 0$. The function $x^2 - a^2$ has critical values at $\pm a$. We calculate a sign table

	$\mathbf{x < -a}$	$\mathbf{x = -a}$	$\mathbf{-a < x < a}$	$\mathbf{x = a}$	$\mathbf{x > a}$
$\mathbf{x - a}$	−	−	−	0	+
$\mathbf{x + a}$	−	0	+	+	+
$\mathbf{x^2 - a^2}$	+	0	−	0	+

and so the solution is $-a \le x \le a$

This result is a precursor of discussions we shall have in the next chapter.

Key Fact 3.4 Square Roots of Inequalities

The following are equivalent (for $a > 0$):

$$\begin{aligned} x^2 \le a^2 &\iff -a \le x \le a \\ x^2 < a^2 &\iff -a < x < a \\ x^2 \ge a^2 &\iff x \le -a \quad \text{or} \quad x \ge a \\ x^2 > a^2 &\iff x < -a \quad \text{or} \quad x > a \end{aligned}$$

Completing the square enables us to solve quadratic inequalities, even if the quadratics do not factorise neatly.

Example 3.8.6. *Solve the inequalities: (a)* $2x^2 - 8x + 11 \le 0$, *(b)* $2x^2 - 10x + 7 \le 0$.

(a) Completing the square tells us that $2x^2 - 8x + 11 = 2(x-2)^2 + 3$. Since it is a square, $(x-2)^2 \ge 0$ for all real x, and so $2x^2 - 8x + 11 \ge 3$ for all real x, so there are no solutions to the inequality $2x^2 - 8x + 11 \le 0$.

(b) Completing the square gives

$$\begin{aligned} 2x^2 - 10x + 7 &\le 0 \\ 2\left(x - \tfrac{5}{2}\right)^2 - \tfrac{11}{2} &\le 0 \\ \left(x - \tfrac{5}{2}\right)^2 &\le \tfrac{11}{4} \end{aligned}$$

and hence $-\frac{1}{2}\sqrt{11} \le x - \frac{5}{2} \le \frac{1}{2}\sqrt{11}$, and so the solution is the interval

$$\tfrac{1}{2}(5 - \sqrt{11}) \le x \le \tfrac{1}{2}(5 + \sqrt{11})$$

The properties of the discriminant can lead to solving quadratic inequalities.

Example 3.8.7. *The quadratic* $2x^2 - kx + 50$ *has* 2 *real roots. What does this tell us about the constant* k*?*

This quadratic must have positive discriminant, and so $k^2 - 400 > 0$, which tells us that either $k > 20$ or $k < -20$.

Example 3.8.8. *Solve the inequality* $\frac{x+3}{5x-1} \le \frac{1}{3}$.

When solving equations of this type, our first instinct is to clear the denominators. This creates a problem, because we do not know whether $5x - 1$ is positive or not; multiplying this inequality by $5x - 1$ will change the direction of the inequality when $x < \frac{1}{5}$. We avoid this problem by multiplying by $(5x - 1)^2$, which is always nonnegative. Note that the original inequality only makes sense if $x \ne \frac{1}{5}$ (otherwise we are being asked to divide by 0), and so we may assume that $(5x - 1)^2 > 0$ for all relevant values of x.

Thus the inequality becomes, after multiplying by $3(5x - 1)^2$,

$$\begin{aligned} 3(x+3)(5x-1) &\le (5x-1)^2 \\ (5x-1)^2 - 3(x+3)(5x-1) &\ge 0 \\ 2(5x-1)(x-5) &\ge 0 \end{aligned}$$

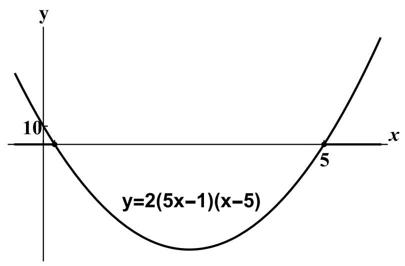


Figure 3.10

According to the sketch, we want the regions $x \le \frac{1}{5}$ or $x \ge 5$. However, we need to remember that the original inequality excluded the possibility of $x = \frac{1}{5}$, and so we must exclude that point from our solution. Thus the solution to the original inequality is either $x < \frac{1}{5}$ or $x \ge 5$.

Note that this technique of multiplying by the square of the denominator automatically creates a factor of $5x - 1$ in what follows. Do not make the mistake of multiplying out both sides of the inequality, since you might lose track of the factor which this method gives us for free!

An alternative approach would be to consider this inequality in cases, allowing for the two possible signs of $5x - 1$. Thus

- If $x > \frac{1}{5}$, the inequality becomes $3(x+3) \le 5x - 1$, or $x \ge 5$.
- If $x < \frac{1}{5}$, the inequality becomes $3(x+3) \ge 5x - 1$, or $x \le 5$.

Putting these two cases together we determine that the set of solutions is $x < \frac{1}{5}$ or $x \ge 5$, as before.

EXERCISE 3F

Solve the following inequalities. Give exact answers. Where necessary, assume that a and b are positive, with $a < b$.

1. *a)* $(x-2)(x-3) < 0$ *b)* $(x-4)(x-7) > 0$ *c)* $(x-1)(x-3) < 0$
   d) $(x-4)(x+10) \geq 0$ *e)* $(2x-1)(x+3) > 0$ *f)* $(3x-2)(3x+5) \leq 0$
   g) $(x+2)(4x+5) \geq 0$ *h)* $(1-x)(3+x) < 0$ *i)* $(3-2x)(5-x) > 0$
   j) $(x-5)(x+5) < 0$ *k)* $(3-4x)(3x+4) > 0$ *l)* $(2+3x)(2-3x) \leq 0$

2. *a)* $x^2+3x-5 > 0$ *b)* $x^2+6x+9 < 0$ *c)* $x^2-5x+2 < 0$
   d) $x^2-x+1 \geq 0$ *e)* $x^2-9 > 0$ *f)* $x^2+2x+1 \leq 0$
   g) $2x^2-3x-1 < 0$ *h)* $8-3x-x^2 > 0$ *i)* $2x^2+7x+1 \geq 0$
   j) $x^2-(a+b)x+ab > 0$ *k)* $x^2+(a-b)x-ab \leq 0$ *l)* $x^2-ab < 0$

3. *a)* $x^2+5x+6 > 0$ *b)* $x^2-7x+12 < 0$ *c)* $x^2-2x-15 \leq 0$
   d) $2x^2-18 \geq 0$ *e)* $2x^2-5x+3 \geq 0$ *f)* $6x^2-5x-6 < 0$
   g) $x^2+5x+2 > 0$ *h)* $7-3x^2 < 0$ *i)* $x^2+ax+a^2 < 0$
   j) $x^2+ax-b < 0$ *k)* $12x^2+5x-3 > 0$ *l)* $3x^2-7x+1 \leq 0$

4. *a)* $\frac{4}{x-7} \leq 1$ *b)* $\frac{12}{x+4} > 2$ *c)* $3+\frac{4}{2x-3} < 0$
   d) $7-\frac{x}{5-2x} \geq 0$ *e)* $\frac{x+2}{7x+4}-2 < 0$ *f)* $7+\frac{5-x}{x} < 0$

5. The quadratic equation $kx^2+6x+k = 0$ has two real roots. What range of values of k are possible?

6. The quadratic equation $kx^2+(k+1)x+2 =$ has no real roots. What range of values of k are possible?

7. The quadratic equation $kx^2+(k+2)x+(2k+1) = 0$ has no real roots. What range of values of k are possible?

8*. Solve the following inequalities, giving exact answers.
   a) $x^3-7x^2+10x < 0$ *b)* $\frac{2+3x}{4-x} \geq 2x$ *c)* $\frac{x+3}{x-3}+\frac{x+1}{(x-2)^2} \leq 0$

Chapter 3: Summary

- Every quadratic expression can be expressed in the form $p(x+q)^2+r$ for some constants p, q and r. The vertex of the quadratic curve $y = p(x+q)^2+r$ occurs at the point $(-q,r)$.
- If $p > 0$, then r is the minimum value of the quadratic $p(x+q)^2+r$, and this value is achieved at $x = -q$. If $p < 0$, then r is the maximum value of the quadratic $p(x+q)^2+r$, and this value is achieved at $x = -q$.
- The quadratic equation $ax^2+bx+c = 0$ can be solved by factorisation, by use of the quadratic formula
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a},$$
or by completing the square.
- The discriminant of the quadratic ax^2+bx+c is the quantity $\Delta = b^2-4ac$. If $\Delta > 0$, the equation $ax^2+bx+c = 0$ has two real roots; if $\Delta = 0$ this equation has one real root; if $\Delta < 0$ this equation has no real roots.
- Inequalities, both quadratic and linear and even simultaneous, are best solved by sketching the corresponding curves and considering their signs.

Review Exercises 1

1. Express $\frac{5}{\sqrt{7}}$ in the form $k\sqrt{7}$ where k is rational. (OCR)

2. In the triangle PQR, Q is a right angle, $PQ = \left(6-2\sqrt{2}\right)$ cm and $QR = \left(6+2\sqrt{2}\right)$ cm. Find the area of the triangle, and show that $PR = 2\sqrt{22}$ cm.

3. Simplify $\sqrt[3]{36} \times \sqrt[6]{\frac{4}{3}} \times \sqrt{27}$.

4. Write $(\sqrt{3})^{-3} + (\sqrt{3})^{-2} + (\sqrt{3})^{-1} + (\sqrt{3})^{0} + (\sqrt{3})^{1} + (\sqrt{3})^{2} + (\sqrt{3})^{3}$ in the form $a + b\sqrt{3}$, where a, b are rational.

5. If $u = x - x^{-1}$ and $v = x^2 + x^{-2}$, find $u(v+1)$ in terms of x, giving your answer in its simplest form.

6. Solve the simultaneous equations $5x - 3y = 41$ and $(7\sqrt{2})x + (4\sqrt{2})y = 82$.

7. Express $(9a^4)^{-\frac{1}{2}}$ as an algebraic fraction in simplified form. (OCR)

8. Show that the triangle formed by the points $(-2,5)$, $(1,3)$ and $(5,9)$ is right-angled.

9. A triangle is formed by the points $A(-1,3)$, $B(5,7)$ and $C(0,8)$.

   a) Show that the angle $\angle ACB$ is a right angle.
   b) Find the coordinates of the point where the line through B parallel to AC cuts the x-axis.

10. $A(7,2)$ and $C(1,4)$ are two vertices of a square $ABCD$.

    a) Find the equation of the diagonal BD.
    b) Find the coordinates of B and of D.

11. A quadrilateral $ABCD$ is formed by the points $A(-3,2)$, $B(4,3)$, $C(9,-2)$ and $D(2,-3)$.

    a) Show that all four sides are equal in length.
    b) Show that $ABCD$ is not a square.

12. P is the point $(7,5)$ and ℓ_1 is the line with equation $3x + 4y = 16$.

    a) Find the equation of the line ℓ_2 which passes through P and is perpendicular to ℓ_1.
    b) Find the point of intersection of the lines ℓ_1 and ℓ_2.
    c) Find the perpendicular distance of P from the line ℓ_1.

13. Prove that the triangle with vertices $(-2,8)$, $(3,20)$ and $(11,8)$ is isosceles. Find its area.

14. Find the equation of the perpendicular bisector of the line joining $(2,-5)$ and $(-4,3)$.

15. The points $A(1,2)$, $B(3,5)$, $C(6,6)$ and D form a parallelogram. Find the coordinates of the midpoint of AC. Use your answer to find the coordinates of D.

16. The point P is the foot of the perpendicular from the point $A(0,3)$ to the line $y = 3x$.
    a) Find the equation of the line AP.
    b) Find the coordinates of the point P.
    c) Find the perpendicular distance of A from the line $y = 3x$.

17. Points which lie on the same straight line are called **collinear**. Show that the points $(-1,3)$, $(4,7)$ and $(-11,-5)$ are collinear.

18. Find the equation of the straight line that passes through the points $(3,-1)$ and $(-2,2)$, giving your answer in the form $ax + by + c = 0$. Hence find the coordinates of the point of intersection of the line and the x-axis. (OCR)

19. The coordinates of the points A and B are $(3,2)$ and $(4,-5)$ respectively. Find the coordinates of the midpoint of AB, and the gradient of AB.

    Hence find the equation of the perpendicular bisector of AB, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. (OCR)

20. The curve $y = 1 + \frac{1}{2+x}$ crosses the x-axis at the point A and the y-axis at the point B.
    a) Calculate the coordinates of A and of B.
    b) Find the equation of the line AB.
    c) Calculate the coordinates of the point of intersection of the line AB and the line with equation $3y = 4x$. (OCR)

21. The straight line p passes through the point $(10,1)$ and is perpendicular to the line r with equation $2x + y = 1$. Find the equation of p.

    Find also the coordinates of the point of intersection of p and r, and deduce the perpendicular distance from the point $(10,1)$ to the line r. (OCR)

22. The line $3x - 4y = 8$ meets the y-axis at A. The point C has coordinates $(-2,9)$. The line through C perpendicular to $3x - 4y = 8$ meets it at B. Calculate the area of the triangle ABC.

23. The points $A(-3,-4)$ and $C(5,4)$ are the ends of the diagonal of a rhombus $ABCD$.
    a) Find the equation of the diagonal BD.
    b) Given that the side BC has gradient $\frac{5}{3}$, find the coordinates of B and hence of D.

24. Two lines have equations $y = m_1x + c_1$ and $y = m_2x + c_2$, where $m_1m_2 = -1$. Prove that the lines are perpendicular.

25. Find the values of x for which $x^{\frac{1}{3}} - 2x^{-\frac{1}{3}} = 1$. (OCR)

26. Solve the equation $4^{2x} \times 8^{x-1} = 32$.

27. Given that $343^n = 49^{n+2}$, find the value of n.

28. Solve the equation $\frac{125^{3x}}{5^{x+4}} = \frac{25^{x-2}}{3125}$.

29. Solve the simultaneous equations $x + y = 2$ and $x^2 + 2y^2 = 11$. (OCR)

30. For what values of k does $2x^2 - kx + 8 = 0$ have a repeated root?

31. A rectangle has perimeter 16 cm and its area is at least 15 cm^2. If one side of the rectangle has length x cm, form a quadratic inequality in x, and find the set of possible values of x.

32. a) Solve the equation $x^2 - (6\sqrt{3})x + 24 = 0$ exactly.
    b) Find all four solutions of the equation $x^4 - (6\sqrt{3})x^2 + 24 = 0$, giving your answers correct to 2 decimal places. (OCR, adapt.)

33. Show that the line $y = 3x - 3$ and the curve $y = (3x+1)(x+2)$ do not meet.

34. Find, correct to 3 significant figures, all the roots of the equation $8x^4 - 8x^2 + 1 = \frac{1}{2}\sqrt{3}$. (OCR)

35. The constant k is real, and the equation $10x^2 + 8x + 1 = k(x^2 + 2x)$ has one real root. Find the range of values of k.

36. The equation of a curve is $y = ax^2 - 2bx + c$, where a, b, c are constants with $a > 0$.

    a) Find, in terms of a, b, c, the coordinates of the vertex of the curve.

    b) Given that the vertex of the curve lies on the line $y = x$, find an expression for c in terms of a and b. Show that, in this case, $4ac \geq -1$, irrespective of the value of b. (OCR, adapt.)

37. Solve the inequality $x(x+1) < 12$. (OCR)

38. Solve the inequality $x - x^3 < 0$.

39. Solve the inequality $x^3 \geq 6x - x^2$.

40. Find the set of values of x for which $9x^2 + 12x + 7 > 19$. (OCR)

41*. Let A and B have coordinates (p_1, q_1) and (p_2, q_2) respectively. Suppose that P is a point with coordinates (x, y). Find the condition that must be satisfied by x and y if $AP = PB$. Hence show that the locus of points equidistant from A and B is the perpendicular bisector of A and B.

42*. Find all real roots of the following equations:

    a) $x + 10\sqrt{x+2} - 22 = 0$,

    b) $x^2 - 4x + \sqrt{2x^2 - 8x - 3} - 9 = 0$,

    giving exact answers. (STEP)

43*. Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y = a(x - y + 2)^2 + b(cx + y)^2 + d$$

find the values of the constants a, b, c and d.

Solve the simultaneous equations (STEP)

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9 \qquad 6x^2 + 3y^2 - 8xy + 8x - 8y = 14\,.$$